Partial Differential Equations

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BY ARNOLD SOMMERFELD

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FOREWORD

The topic with which I regularly conclude my six-term series of lectures in Munich is the partial differential equations of physics. We do not really deal with mathematical physics, but with *physical mathematics;* not with the mathematical formulation of physical facts, but with the physical motivation of mathematical methods. The oft-mentioned "prestabilized harmony" between what is mathematically interesting and what is physically important is met at each step and lends an esthetic — I should like to say metaphysical -- attraction to our subject.

The problems to be treated belong mainly to the classical mathematical literature, as shown by their connection with the names of Laplace, Fourier, Green, Gauss, Riemann, and William Thomson. In order to show that these methods are adequate to deal with actual problems, we treat the *propagation of radio waves* in some detail in Chapter VI.

Chapter V deals with the general method of *eigenfunctions*. The most spectacular domain of application of that method is *wave mechanics*, as we show here with the help of some selected, particularly simple examples. The mathematically rigorous foundation of the existence and the properties of eigenfunctions with the help of theorems about integral equations cannot be given here; the latter are mentioned only occasionally as the counterpart of the corresponding theorems on differential equations.

Chapter IV on *Bessel functions* and *spherical harmonics* is comparatively lengthy despite a development that is as concise as possible. For the sake of brevity we have relegated some proofs to the exercises, as we have also done in other chapters. A special section is dedicated to the beautiful *method of reciprocal radii* and to the demonstration of the fact that it unfortunately cannot be applied to other than potential problems.

Chapter III deals exclusively with the classic problem of heat conduction. In addition to the Fourier method we develop in detail the intuitive method of *reflected images* for regions with plane boundaries. Chapter II deals with the different types of differential equations and boundary value problems; *Green's theorem* and *Green's function* are introduced in considerable generality.

Chapter I about Fourier series and integrals is based throughout on the *method of least squares*. If the latter is complemented by a requirement which we called "the condition of finality," then we can replace the more formal computations of the older developments in a complete and generalizable way, not only in the trigonometric case but also for spherical harmonics and general eigenfunctions.

As is seen from this survey, the arrangement of the material is determined not by systematic but by didactic points of view. Chapter I intends to put the reader in the midst of the methodology of the Fourier and the Fourier-like expansions. Only in Chapter II do we start to introduce the concepts from the theory of partial differential equations that are of the greatest importance for the mathematical physicist. From a systematic point of view Chapter III would be subordinated to the general methods of Chapter V but it precedes it for historic and didactic reasons. The lengthiness of Chapter IV may be justified by the fact that a large part of the material contained in the textbooks on Bessel functions and spherical harmonics is at least sketched there, and is put in readiness for application. The formal mathematical part is interrupted for didactic reasons for both classes of functions by typical examples of applications.

It is obvious that this material could not be presented completely in a short summer term. In fact several mathematically more complicated sections have been added in print, some of these in the form of appendixes. In this connection we wish to mention Appendix II to Chapter V, which was added only after the completion of the rest of the manuscript and which is likely to be of fundamental importance for problems dealing with the intermittent range between short waves and long waves, that is, for the passage from geometrical optics to wave optics.

In the preparation of the manuscript I was able to rely on the lecture notes of R. Schlatterer for 1935, as well as on earlier notes of Professor J. Meixner. My friend F. Sauter critically perused the entire manuscript and has also been most generous in giving me his own improved version on many points. I owe him more than I can point out in the text. My colleague, J. Lense, examined the manuscript from the mathematical point of view. Dr. F. Renner collaborated on the last chapter especially; H. Schmidt advised me on the arrangement of the material.

ARNOLD SOMMERFELD.

[*Publisher's note:* This is a translation of Sommerfeld's "Lectures on Theoretical Physics," Volume VI. Translations of Volume I entitled, "Mechanics," and Volume II entitled, "Mechanics of Deformable Bodies," are in preparation. In this text they are referred to as v. I and v. II.]

EDITORS' FOREWORD

This book is the first volume in a projected new series of mathematical books to appear under the title "Pure and Applied Mathematics." The books of the new series will be "advanced" in the sense that they will maintain a standard of scientific maturity. It is not intended, however, to adhere to any rigid pattern of presentation or degree of difficulty. Thus there will be a place for textbooks for first-year graduate students as well as monographs for research workers and possibly an occasional treatise. It is the hope of the Editors that these volumes will find a worthy place in the growing list of excellent scientific works which have appeared in recent years.

P. A. S.
S. E.

New York, 1949.

ERRATUM

"Eigenvalues" (see pp. 166ff.) should be written as one word. The two-word form is incorrect.

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CHAPTER I

Fourier Series and Integrals

Fourier's Théorie analytique de la chaleur[1] is the bible of the mathematical physicist. It contains not only an exposition of the trigonometric series and integrals named after Fourier, but the general boundary value problem is treated in an exemplary fashion for the typical case of heat conduction.

In mathematical lectures on Fourier series emphasis is usually put on the concept of arbitrary function, on its continuity properties and its singularities (accumulation points of an infinity of maxima and minima). This point of view becomes immaterial in the physical applications. For, the initial or boundary values of functions considered here, partially because of the atomistic nature of matter and of interaction, must always be taken as smoothed mean values, just as the partial differential equations in which they enter arise from a statistical averaging of much more complicated elementary laws. Hence we are concerned with relatively simple idealized functions and with their approximation with "least possible error." What is meant by the latter is explained by Gauss in his "Method of Least Squares." We shall see that it opens a simple and rigorous approach not only to Fourier series but to all other series expansions of mathematical physics in spherical and in cylindrical harmonics, or generally in eigenfunctions.

§ 1. Fourier Series

Let an arbitrary function $f(x)$ be given in the interval $-\pi \leq x \leq +\pi$; this function may, e.g., be an empirical curve determined by sufficiently many and sufficiently accurate measurements. We want to approximate it by the sum of $2n + 1$ trigonometric terms

$$S_n(x) = A_0 + A_1 \cos x + A_2 \cos 2x + \cdots + A_n \cos nx + B_1 \sin x + B_2 \sin 2x + \cdots + B_n \sin nx \tag{1}$$

[1] Jean Baptiste Fourier, 1768–1830. His book on the conduction of heat appeared in 1822 in Paris. Fourier also distinguished himself as an algebraist, engineer, and writer on the history of Egypt, where he had accompanied Napoleon.

The influence of his book even outside France is illustrated by the following quotation: "Fourier's incentive kindled the spark in (the then 16-year-old) William Thomson as well as in Franz Neumann." (F. Klein, Vorlesungen über die Geschichte der Mathematik im 19. Jahrhundert, v. I, p. 233.)

By what criterion shall we choose the coefficients A_k, B_k at our disposal? We shall denote the error term $f(x) - S_n(x)$ by $\varepsilon_n(x)$; thus

$$f(x) = S_n(x) + \varepsilon_n(x). \tag{2}$$

Following Gauss we consider the *mean square error*

$$M = \frac{1}{2\pi} \int_{-\pi}^{+\pi} \varepsilon_n^2 \, dx \tag{3}$$

and reduce M to a minimum through the choice of the A_k, B_k.

To this we further remark that the corresponding measure of the total error formed with the first power of ε_n would not be suitable, since arbitrarily large positive and negative errors could then cancel each other and would not count in the total error. On the other hand the use of the absolute value $|\varepsilon_n|$ under the integral sign in place of ε_n^2 would be inconvenient because of its non-analytic character.[2]

The requirement that (3) be a minimum leads to the equations

$$\begin{aligned} -\frac{\partial M}{\partial A_k} &= \frac{1}{\pi} \int_{-\pi}^{+\pi} \{f(x) - S_n(x)\} \cos kx \, dx = 0, \quad k = 0, 1, 2, \ldots, n \\ -\frac{\partial M}{\partial B_k} &= \frac{1}{\pi} \int_{-\pi}^{+\pi} \{f(x) - S_n(x)\} \sin kx \, dx = 0, \quad k = \quad 1, 2, \ldots, n. \end{aligned} \tag{4}$$

These are exactly $2n + 1$ equations for the determination of the $2n + 1$ unknowns A, B. A favorable feature here is that each individual coefficient A or B is determined directly and is not connected recursively with the other A, B. We owe this to the *orthogonality relations* that exist among trigonometric functions:[3]

$$\int \cos kx \sin lx \, dx = 0, \tag{5}$$

$$\left.\begin{aligned} &\int \cos kx \cos lx \, dx \quad (5a) \\ &\int \sin kx \sin lx \, dx \quad (5b) \end{aligned}\right\} = 0, \; k \neq l.$$

[2] A completely different approach is taken by the great Russian mathematician Tchebycheff in the approximation named after him. He considers not the *mean* but the *maximal* $|\varepsilon_n|$ appearing in the interval of integration, and makes this a minimum through the choice of the coefficients at his disposal.

[3] Here and below all integrals are to be taken from $-\pi$ to $+\pi$. In order to justify the word "orthogonality" we recall that two vectors **u**,**v** which are orthogonal in Euclidean three dimensional, or for that matter n-dimensional space, satisfy the condition that their scalar product

In order to prove them it is not necessary to write down the cumbersome addition formulae of trigonometric functions, but to think rather of their connection with the exponential functions $e^{\pm ikx}$ and $e^{\pm ilx}$. The integrands of (5a,b) consist then of only four terms of the form $\exp\{\pm i(k+l)x\}$ or $\exp\{\pm i(k-l)x\}$, all of which vanish upon integration unless $l = k$. This proves (5a,b). The fact that (5) is valid even without this restriction follows from the fact that for $l = k$ it reduces to

$$\frac{1}{4i}\int_{-\pi}^{\pi}(e^{2ikx}-e^{-2ikx})\,dx = 0$$

In a similar manner one obtains the values of (5a,b) for $l = k > 0$ (only the product of $\exp(ikx)$ and $\exp(-ikx)$ contributes to it): this value simply becomes equal to π; for $l = k = 0$ the value of the integral in (5a) obviously equals 2π. We therefore can replace (5a,b) by the single formula which is valid also for $l = k > 0$

$$\text{(6)}\qquad \frac{1}{\pi}\int \cos kx\cos lx\,dx = \frac{1}{\pi}\int \sin kx\sin lx\,dx = \delta_{kl}$$

with the usual abbreviation

$$\delta_{kl} = \begin{cases} 0 \ldots l \neq k \\ 1 \ldots l = k > 0. \end{cases}$$

Equation (6) for $k = l$ is called the *normalizing condition*. It is to be augmented for the exceptional case $l = k = 0$ by the trivial statement

$$\text{(6a)}\qquad \frac{1}{2\pi}\int dx = 1.$$

If we now substitute (5),(6) and (6a) in (4) then in the integrals with S_n all terms except the k-th vanish, and we obtain directly *Fourier's representation of coefficients:*

$$\text{(7)}\qquad \begin{aligned} &\left.\begin{aligned} A_k &= \frac{1}{\pi}\int f(x)\cos kx\,dx \\ B_k &= \frac{1}{\pi}\int f(x)\sin kx\,dx \end{aligned}\right\} k > 0. \\ &A_0 = \frac{1}{2\pi}\int f(x)\,dx. \end{aligned}$$

$$(\mathbf{u}\,\mathbf{v}) = \sum_{1}^{N} \mathbf{u}_i\,\mathbf{v}_i = 0$$

vanish. The integrals appearing in (5) can be considered as sums of this same type with infinitely many terms. See the remarks in §26 about so-called "Hilbert space."

Our approximation S_n is hereby determined completely. If, e.g. $f(x)$ were given empirically then the integrations (7) would have to be carried out numerically or by machine.[4]

From (7) one sees directly that for an even function $f(-x) = f(+x)$, all B_k vanish, whereas for an odd function, $f(-x) = -f(+x)$, all A_k, including A_0, vanish. Hence the former is approximated by a *pure cosine series,* the latter by a *pure sine series.*

The accuracy of the approximation naturally increases with the number of constants A,B at our disposal, i.e., with increasing n. Here the following fortunate fact should be stressed: since the A_k,B_k for $k<n$ are independent of n, the previously calculated A_k,B_k remain unchanged by the passage from n to $n+1$, and only the coefficients A_{n+1}, B_{n+1} have to be newly calculated. The A_k,B_k, once found, are *final.*

There is nothing to prevent us from letting n grow indefinitely, that is, to perform the passage to the limit $n \to \infty$. The finite series considered so far thereby goes over into an *infinite Fourier series.* The following two sections will deal with its convergence.

More complicated than the question of convergence is that of the *completeness* of the system of functions used here as basis. It is obvious that if in the Fourier series one of the terms, e.g., the k-th cosine term, were omitted, then the function $f(x)$ could no longer be described by the remaining terms with arbitrary accuracy; even in passing to the limit $n \to \infty$ a finite error $A_k \cos kx$ would remain. To take an extremely simple case, if one attempted to express $\cos nx$ by an incomplete series of all cosine terms with $k<n$ and $k>n$, then all A_k would vanish because of orthogonality and the error would turn out to be $\cos nx$ itself. Of course it would not occur to anyone to disturb the regularity of a system like that of the trigonometric functions by the omission of one term. But in more general cases such considerations of mathematical esthetics need not be compelling.

What the mathematicians teach us on this question with their *relation of completeness* is in reality no more than is contained in the basis of the method of least squares. One starts, namely, with the remark that a system of functions say $\varphi_0, \varphi_1, \ldots, \varphi_k, \ldots,$ can be complete only if *for every continuous function* $f(x)$ the mean error formed according to (3) goes *to zero* in the limit $n \to \infty$. It is assumed that the system of φ is orthogonal and normalized to 1, that is

$$\int \varphi_k \varphi_l \, dx = 0, \qquad \int \varphi_k^2 \, dx = 1, \tag{8}$$

[4] Integrating machines that serve in Fourier analysis are called "harmonic analyzers." The most perfect of these is the machine of Bush and Caldwell; it can be used also for the integration of arbitrary simultaneous differential equations; see *Phys. Rev.* **38**, 1898 (1931).

which implies that the expansion coefficients A_k are simply

$$A_k = \int f(x)\,\varphi_k(x)\,dx. \tag{9}$$

Let the limits of integration in this and the preceding integrals be a and b so that the length of the interval of expansion is $b - a$. One then forms according to (3)

$$(b-a)\,M = \int \Big(f - \sum_{k=0}^{n} A_k\varphi_k\Big)^2 dx = \int f^2\,dx - 2\sum_{k=0}^{n} A_k \int f\varphi_k\,dx + \sum_{k=0}^{n} A_k^2.$$

Equation (8) has been used in the last term here. According to (9) the middle term equals twice the last term except for sign. Hence

$$\operatorname*{Lim}_{n\to\infty}(b-a)\,M = \int f^2\,dx - \sum_{k=0}^{\infty} A_k^2$$

and one requires, as remarked above, that for every continuous function

$$\sum A_k^2 = \int f^2\,dx. \tag{10}$$

This is the mathematical formulation of the *relation of completeness* which is so strongly emphasized in the literature. It is obvious that it can hardly be applied as a practical criterion. Also, since it concerns only the mean error, it says nothing on the question of whether the function f is really represented everywhere by the Fourier series (see also §3, p. 15).

In this introductory section we have followed the historical development in deducing the *finality of the Fourier coefficients* from the *orthogonality of the trigonometric functions*. In §4 we shall demonstrate, for the typical case of spherical harmonics, that, conversely, orthogonality can be deduced quite generally from our requirement of *finality*. From our point of view of approximation this seems to be the more natural approach. In any case it should be stressed at this point that *orthogonality* and *requirement of finality* imply each other and can be replaced by each other.

Finally, we want to translate our results into a form that is mathematically more perfect and physically more useful. We carry this out for the case of infinite Fourier series, remarking however, that the following is valid also for a truncated series — actually the more general and rigorous case.

We write, replacing the variable of integration in (7) by ξ:

$$
\begin{aligned}
f(x) &= \frac{1}{2\pi}\int f(\xi)\,d\xi + \\
&\quad \frac{1}{\pi}\sum_{k=1}^{\infty}\int f(\xi)\cos k\xi\,d\xi\cdot\cos kx + \frac{1}{\pi}\sum_{k=1}^{\infty}\int f(\xi)\sin k\xi\,d\xi\cdot\sin kx \\
&= \frac{1}{2\pi}\int f(\xi)\,d\xi + \frac{1}{\pi}\sum_{k=1}^{\infty}\int f(\xi)\cos k(x-\xi)\,d\xi \\
&= \frac{1}{2\pi}\left\{\int f(\xi)\,d\xi + \sum_{k=1}^{\infty}\left(\int f(\xi)\,e^{ik(x-\xi)}d\xi + \int f(\xi)\,e^{-ik(x-\xi)}d\xi\right)\right\}.
\end{aligned}
$$

In the last term we can consider the summation for positive k in $\exp\{-ik(x-\xi)\}$ to be the summation for the corresponding negative values of k in $\exp\{+ik(x-\xi)\}$. We therefore replace this term by

$$\sum_{k=-1}^{-\infty}\int f(\xi)\,e^{ik(x-\xi)}\,d\xi = \sum_{k=-\infty}^{-1}\int f(\xi)\,e^{ik(x-\xi)}\,d\xi\,.$$

Then the uncomfortable exceptional position of the term $k = 0$ is removed: it now fits between the positive and negative values of k and we obtain

$$f(x) = \frac{1}{2\pi}\sum_{k=-\infty}^{+\infty}\int f(\xi)\,e^{ik(x-\xi)}\,d\xi\,. \tag{11}$$

Finally, introducing the Fourier coefficients C_k, which are *complex* even for real $f(x)$:

$$f(x) = \sum_{k=-\infty}^{+\infty} C_k e^{ikx},\quad C_k = \frac{1}{2\pi}\int f(\xi)\,e^{-ik\xi}\,d\xi. \tag{12}$$

The relations among the C's and the A's and B's defined by (7), are given by

$$C_k = \begin{cases} \frac{1}{2}(A_k - i\,B_k), & k > 0, \\ \frac{1}{2}(A_{|k|} + i\,B_{|k|}), & k < 0, \end{cases} \tag{13}$$

$$C_0 = A_0\,.$$

Our complex representation (12) is obviously simpler than the usual real representation; it will be of special use to us in the theory of Fourier integrals.

If we extend our representation, originally intended for the interval

$-\pi < x < +\pi$, to the intervals $x > \pi$ and $x < -\pi$ then we obtain continued periodic repetitions of the branch between $-\pi$ and $+\pi$; in general they do not constitute the analytic continuation of our original function $f(x)$. In particular the periodic function, thus obtained will have *discontinuities* for the odd multiples of $\pm\pi$, unless we happen to have $f(-\pi) = f(+\pi)$. The next section deals with the investigation of the error arising at such a point.

§ 2. Example of a Discontinuous Function. Gibbs' Phenomenon and Non-Uniform Convergence

Let us consider the function

(1) $$f(x) = \begin{cases} +1 \text{ for } 0 < x < \pi \\ -1 \text{ for } -\pi < x < 0. \end{cases}$$

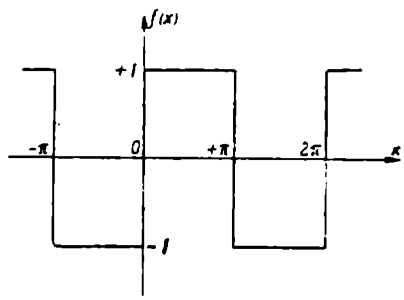


Fig. 1. The chain of segments $y = \pm 1$ for positive and negative $|x| < \pi$ and its periodic repetition represented by the Fourier series.

We sketch it in Fig. 1 with its periodic repetitions completed by the vertical connecting segments of length 2 at the points of discontinuity $x = 0, \pm\pi, \pm 2\pi, \ldots$, whereby it becomes a "meander line." Our function f is odd, its Fourier series consists therefore solely of sine terms as pointed out in (1.7). The coefficients can best be calculated from equation (1.12), which yields

(1a) $$\begin{aligned} C_k &= \frac{1}{2\pi}\left(\int_0^{\pi} e^{-ik\xi}\,d\xi - \int_{-\pi}^{0} e^{-ik\xi}\,d\xi\right) \\ &= \frac{1}{2\pi}\left(\frac{e^{-ik\pi}-1}{-ik} - \frac{1-e^{+ik\pi}}{-ik}\right) = \frac{(-1)^k - 1}{-i\pi k} = \begin{cases} -\dfrac{2i}{\pi k} \cdots k \text{ odd} \\ 0 \ldots k \text{ even} \end{cases} \end{aligned}$$

This implies according to (1.13):

$$B_k = \frac{4}{\pi k}, \quad k = 1, 3, 5, \ldots$$

We obtain the following sine series:

(2) $$f(x) = \frac{4}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right).$$

One may imagine the upheaval caused by this series when it was first constructed by Fourier. A discontinuous chain formed through the superposition of an infinite sequence of only the simplest continuous functions! Without exaggeration one may say that this series has contributed greatly to the development of the general concept of real function. We shall see below that it also served to deepen the concept of convergence of series.

In order to understand how the series manages to approximate the discontinuous sequence of steps, we draw[5] in Fig. 2 the approximating functions S_1, S_3, S_5 defined by (1.1) together with $S_\infty = f(x)$.

$$S_1 = \frac{4}{\pi}\sin x, \quad S_3 = \frac{4}{\pi}\left(\sin x + \frac{1}{3}\sin 3x\right),$$
$$S_5 = \frac{4}{\pi}\left(\sin x + \frac{1}{3}\sin 3\,x + \frac{1}{5}\sin 5\,x\right).$$

S_1 has its maximum value

$$y = 4/\pi = 1.27,$$

at $x = \pi/2$, and hence rises 27% above the horizontal line $y = 1$, which is to be described. S_3 has a minimum value at the same point and hence

$$y = \frac{4}{\pi}\left(1 - \frac{1}{3}\right) = 0.85\,,$$

stays 15% below the straight line to be described. In addition S_3 also has maxima at $\pi/4$ and $3\pi/4$, which lie 20% above that line. (The reader is invited to check this!) S_5 on the other hand has a maximum of

$$y = \frac{4}{\pi}\left(1 - \frac{1}{3} + \frac{1}{5}\right) = 1.10\,,$$

at $x = \pi/2$ which is too high by only 10%. A flat minimum on either side is followed by two steeper maxima situated near $x = 0$ and $x = \pi$. In general the maxima and minima of S_{2n+1} lie *between* those of S_{2n-1}(see exercise I.1).

All that has been said here about the stepwise approximation of the line $y = +1$, is of course equally valid for its mirror image $y = -1$. It too is approximated by *successive oscillations*, so that the approximating curve S_n swings n times above and $n + 1$ times below the line segment which is to be represented. The oscillations in the *middle part* of the line segment decrease with increasing n; at the *points of discontinuity*

[5] In the lectures at this point abundant use was made of colored chalk. Since this unfortunately is impossible in print, both S_∞ and the approximation S_5, which are the most important for us, are drawn in bolder lines.

$x = 0, \pm\pi, \pm 2\pi, \ldots$, where there is no systematic decrease of the maxima, the approximating curves approach the vertical jumps of discontinuity. The picture of an approximating curve of very large n therefore looks the way it has been pictured schematically in Fig. 3.

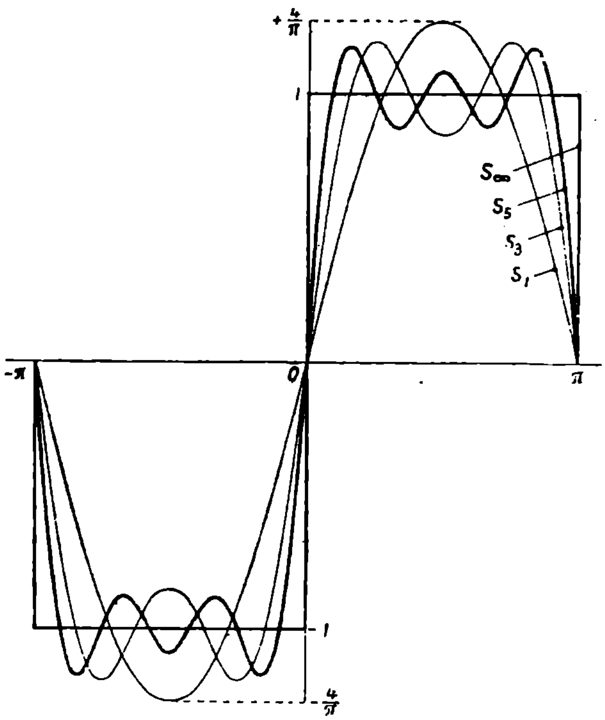


Fig. 2. The approximations of the chain S_∞; the maxima and minima lie at equally spaced values of x, respectively between those of the preceding approximation.

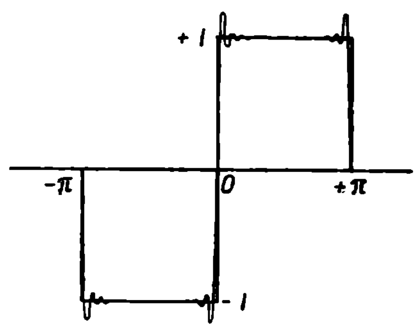


Fig. 3. An approximation S of very high order for the illustration of Gibbs' phenomenon.

We now consider more closely the behavior of $S_{2n+1}(x)$ for large n at one of the jumps, e.g., for $x = 0$. To this end we rewrite the original formula for S_{2n+1} in integral form (an integral usually being easier to discuss than a sum). This is done in the following steps:

$$S_{2n+1} = \frac{4}{\pi} \sum_{k=0}^{n} \frac{\sin(2k+1)x}{2k+1} = \frac{4}{\pi} \sum_{k=0}^{n} \int_0^x \cos(2k+1)\,\xi\, d\xi$$

$$= \frac{2}{\pi} \int_0^x \left\{ \sum_{k=0}^{n} e^{(2k+1)i\xi} + \sum_{k=0}^{n} e^{-(2k+1)i\xi} \right\} d\xi .$$

After factoring out $\exp(\pm i\xi)$ from the two sums of the last line they become geometric series in increasing powers of $\exp(\pm 2i\xi)$ which can be summed in the familiar manner. Therefore, one obtains

$$(3) \qquad S_{2n+1} = \frac{2}{\pi}\int_0^x \left\{e^{i\xi}\frac{1-e^{2i(n+1)\xi}}{1-e^{2i\xi}} + e^{-i\xi}\frac{1-e^{-2i(n+1)\xi}}{1-e^{-2i\xi}}\right\}d\xi .$$

By further factorization these two fractions can be brought to the common form (except for the sign of i):

$$(3\text{a}) \qquad e^{\pm in\xi}\frac{\sin(n+1)\xi}{\sin\xi} .$$

In this way (3) goes over into

$$(3\text{b}) \qquad S_{2n+1} = \frac{2}{\pi}\int_0^x \frac{2\cos(n+1)\xi\ \sin(n+1)\xi}{\sin\xi}\, d\xi .$$

Finally for sufficiently small x we can replace $\sin\xi$ in the denominator by ξ; the corresponding simplification in the numerator would not be permissible since ξ there is accompanied by the large factor $n+1$. We obtain therefore for (3a), if we introduce the new variable of integration u and the new argument v,

$$(4) \qquad S_{2n+1} = \frac{2}{\pi}\int_0^v \frac{\sin u}{u}\, du \ldots \qquad \begin{cases} u = 2(n+1)\xi, \\ v = 2(n+1)x. \end{cases}$$

From this the following conclusion may be drawn: if for finite n we set $x = 0$ then v vanishes and $S_{2n+1} = 0$. If now we allow n to increase toward infinity, the relation $S_{2n+1} = 0$ holds in the limit. Hence

$$(4\text{a}) \qquad \lim_{n\to\infty}\ \lim_{x\to 0} S_{2n+1} = 0 .$$

But if for $x > 0$ we first allow n to approach infinity, then v becomes infinite, and, according to a fundamental formula that will be treated in exercise I.5: $S_{2n+1} = 1$. If we then allow x to decrease towards zero, the value $S_{2n+1} = 1$ holds also for the limit $x = 0$; hence

$$(4\text{b}) \qquad \lim_{x\to 0}\ \lim_{n\to\infty} S_{2n+1} = 1 .$$

The two limiting processes therefore are not interchangeable. If the function $f(x)$ to be represented were continuous at the point $x = 0$, then the order of passage to the limit would be immaterial, and in contrast to (4a,b) one would have

$$\lim_{n\to\infty}\lim_{x\to 0} S_{2n+1} = \lim_{x\to 0}\lim_{n\to\infty} S_{2n+1} = f(v). \tag{4c}$$

This, however, does not exhaust by any means the peculiarities contained in equation (4); in order to develop them we introduce the frequently tabulated[6] "integral sine"

$$Si(v) = \int_0^v \frac{\sin u}{u}\, du \tag{5}$$

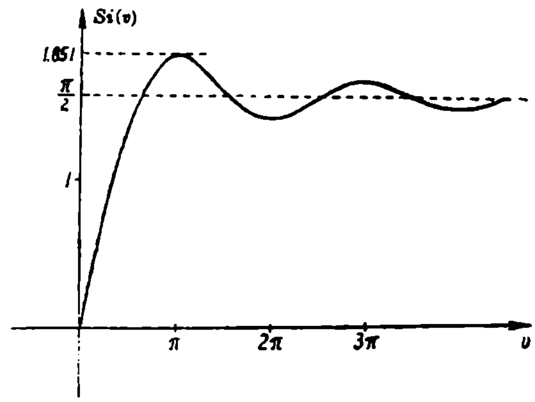


Fig. 4. Graphic representation of the integral sine.

and represent its general form in Fig. 4. It can be described as follows: for small values of v, where $\sin u$ can be set equal to u, we have *proportionality with v;* for large values of v we have *asymptotic approach to* $\pi/2$; in between we have successively *decreasing oscillations* with maxima and minima at $v = \pi, 2\pi, 3\pi, \ldots$, as can be seen from (5); the ordinate of the first and greatest maximum is 1.851 according to the above mentioned tables. To the associated abscissa of the *Si*-curve there corresponds in the original variable x, owing to the relation $v = 2(n+1)x$, the infinite sequence of points

$$\ldots, x_n = \frac{\pi}{2(n+1)}, \quad x_{n+1} = \frac{\pi}{2(n+2)}, \ldots \tag{6}$$

at which according to (4) the approximations $S_{2n+1}, S_{2n+3}, \ldots$ have the fixed value:

$$S = \frac{2}{\pi} 1{,}851 = 1.18. \tag{7}$$

This value, which exceeds $y = 1$ by 18%, is at the same time the *upper limit* of the range of values given by our approximations. Its *lower limit*, $S = -1.18$ is assumed when we approach zero from the negative side in the sequence of points $-x_n, -x_{n+1}, \ldots$. Each point of the range

[6] E.g. B. Jahnke-Emde, Funktionentafeln, Teubner, Leipzig, 3d edition, 1938.

between -1.18 and $+1.18$ can be obtained by a special manner of passing to the limit; e.g., the points $S = 0$ and $S = 1$ are obtained in the manner described in (4a) and (4b).

This behavior of the approximating functions, in particular the appearance of an excess over the range of discontinuity ± 1, is called *Gibbs' phenomenon.* (Willard Gibbs, 1844 to 1906, was one of America's greatest physicists, and simultaneously with Boltzmann, was the founder of statistical mechanics.) Gibbs' phenomenon appears wherever a discontinuity is approximated. One then speaks of the *non-uniform convergence* of the approximation process.

We still want to convince ourselves that actually every point between $S = 1.18$ and $S = -1.18$ can be obtained if we *couple* the two passages to the limit in a suitable fashion. According to (6), this coupling consists in setting $x(n+1)$ or, what comes to the same thing, setting x_n equal to the fixed value $\pi/2$. If instead we take the more general value, q, then from (4) we obtain $v = 2q$, and (4) and (5) together yield

$$S_{2n+1} = \frac{2}{\pi} Si(2q),$$

where $Si(2q)$ can assume all values between 0 and 1.851 with varying positive q, as can be seen directly from Fig. 4. Correspondingly for negative q one obtains all values between 0 and -1.851. The passages to the limit that have thus been coupled yield not only the approach of our approximating function to the discontinuity from -1 to $+1$, but also an excess beyond it, i.e., Gibbs' phenomenon.

In addition to these basic statements we want to deduce some formal mathematical facts from our Fourier representation (2). In particular we substitute $x=\pi/2$ therein and obtain the famous *Leibniz series*

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots \tag{8}$$

This series converges slowly; we obtain more rapidly convergent representations for the powers of π if we integrate (2) repeatedly. For the following refer to Fig. 5 below.

By restricting ourselves to the interval $0 < x < \pi$, we write

$$\frac{\pi}{4} = \sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots \tag{9}$$

instead of (2). Integration from 0 to x yields:

$$\frac{\pi}{4}x = 1 - \cos x + \frac{1}{3^2}(1 - \cos 3x) + \frac{1}{5^2}(1 - \cos 5x) + \cdots \tag{10}$$

Hence for $x = \pi/2$

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots \tag{11}$$

Subtracting (10) from (11) we get:

$$\frac{\pi}{4}\left(\frac{\pi}{2} - x\right) = \cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \cdots \tag{12}$$

By another integration from 0 to x this becomes

$$\frac{\pi}{8}(\pi x - x^2) = \sin x + \frac{1}{3^3}\sin 3x + \frac{1}{5^3}\sin 5x + \cdots \tag{13}$$

Hence for $x = \pi/2$, as an *analogue to the Leibniz series*

$$\frac{\pi^3}{32} = 1 - \frac{1}{3^3} + \frac{1}{5^3} - \cdots \tag{14}$$

We integrate (13) once more with respect to x and set $x = \pi/2$:

$$\frac{\pi}{8}\left(\pi\frac{x^2}{2} - \frac{x^3}{3}\right) = 1 - \cos x + \frac{1}{3^4}(1 - \cos 3x) + \frac{1}{5^4}(1 - \cos 5x) + \cdots \tag{15}$$

$$\frac{\pi^4}{3 \cdot 32} = 1 - \frac{1}{3^4} + \frac{1}{5^4} + \cdots \tag{16}$$

Finally we subtract (15) from (16) and have

$$\frac{\pi}{8}\left(\frac{\pi^3}{12} - \frac{\pi x^2}{2} + \frac{x^3}{3}\right) = \cos x + \frac{1}{3^4}\cos 3x + \frac{1}{5^4}\cos 5x + \cdots \tag{17}$$

The series (11) and (16) range only over the *odd* numbers. The series ranging over the *even* numbers are respectively equal to 1/4 and 1/16 of the sums ranging over *all* integers. If we denote the latter by Σ_2 and Σ_4 respectively, then we have

$$\frac{\pi^2}{8} + \frac{1}{4}\sum\nolimits_2 = \sum\nolimits_2 \quad \text{and} \quad \frac{\pi^4}{3 \cdot 32} + \frac{1}{16}\sum\nolimits_4 = \sum\nolimits_4,$$

hence

$$\sum\nolimits_2 = \frac{\pi^2}{6} \quad \text{and} \quad \sum\nolimits_4 = \frac{\pi^4}{90}. \tag{18}$$

This value of Σ_4 was needed in the derivation of Stefan's law of radiation or Debye's law for the energy content of a fixed body. The trigonometric series (12), (13), (17) will be useful examples in the following sections. The higher analogues to the "Leibniz series" (8) and (14) as well as those to Σ_2 and Σ_4 will be computed in exercise I.2.

§ 3. On the Convergence of Fourier Series

We are going to prove the following theorem: If a function $f(x)$, together with its first $n-1$ derivatives is continuous and differentiable between $-\pi$ and $+\pi$ inclusive, and the n-th derivative, is differentiable over the same interval except possibly at a finite number of points $x = x_l$ where it may have bounded discontinuities (i.e., finite jumps), then the coefficients A_k, B_k of its Fourier expansion approach zero at least as fast as k^{-n-1} as $k \to \infty$.

The stipulation "inclusive" in referring to the boundaries of the interval has here the following meaning: every function which is represented by a Fourier series is periodic in nature. An adequate picture of its argument would therefore not be the straight line segment from $-\pi$ to $+\pi$, but a circle closing at $x = \pm\pi$. It is this fact to which the *continuity* of f and its first $n-1$ derivatives at the point $x = \pm\pi$ refers. This point is in no way distinguished from the interior points of the interval, just as it is immaterial whether we denote the boundaries of the interval by $-\pi, +\pi$ or, e.g., by $\frac{\pi}{4}, \frac{9\pi}{4}$ etc.

For the proof of this theorem it is convenient to use the complex form (1.12)

$$(1) \qquad f(x) = \sum_{-\infty}^{+\infty} C_k e^{ikx}, \qquad\qquad (1a) \qquad 2\pi C_k = \int_{-\pi}^{+\pi} f(\xi) e^{-ik\xi} d\xi$$

From (1a) one obtains through integration by parts

$$(2) \qquad 2\pi C_k = \frac{1}{-ik} f(\xi) e^{-ik\xi} \Big|_{-\pi}^{+\pi} + \frac{1}{ik} \int_{-\pi}^{+\pi} f'(\xi) e^{-ik\xi} d\xi .$$

Here the first term on the right side vanishes because of the postulated continuity of f; the second term can again be transformed by integration by parts. After n iterations of the same process one obtains

$$(3) \qquad 2\pi (ik)^n C_k = \int_{-\pi}^{+\pi} f^{(n)}(\xi) e^{-ik\xi} d\xi .$$

Because of the discontinuities of $f^{(n)}(x)$ at $x = x_l$, this integral has to be divided into partial integrals between $x = x_l$ and $x = x_{l+1}$; let the jumps of $f^{(n)}$ at the points of discontinuity be denoted by Δ_l^n. Equation (3) written explicitly then reads:

$$(3a) \qquad 2\pi(ik)^n C_k = \sum_l \int_{x_l}^{x_{l+1}} f^{(n)}(\xi)\, e^{-ik\xi} d\xi,$$

where the point $x = \pm\pi$ may be contained among the points $x = x_l$ By one more partial integration (3a) becomes

$$(4) \qquad 2\pi(ik)^n C_k = \frac{1}{-ik}\sum_l \Delta_l^n e^{-ikx_l} + \frac{1}{ik}\sum_l \int_{x_l}^{x_{l+1}} f^{(n+1)}(\xi)\, e^{-ik\xi} d\xi.$$

Considering the fact that the discontinuities Δ_l^n were assumed to be bounded and that $f^{(n)}$ was assumed to be differentiable between the points of discontinuity, one sees from (4) that C_k vanishes at least to the same order as k^{-n-1} when one lets $k \to \infty$. For special relations between the Δ_l^n or for special behavior of $f^{(n+1)}(\xi)$, the order of vanishing could become even higher.

This theorem is valid for negative k too. This implies that it is valid also for the *real Fourier coefficients* A_k, B_k $(k > 0)$, since according to (1.13) they are expressible in terms of the C_k with positive and negative k.

A special consequence of our theorem is that an *analytic function* of period 2π (such a function is continuous and periodic together with all its derivatives) has Fourier coefficients that decrease faster than any power of $1/k$ with increasing k. An example of this would be an arbitrary polynomial in $\sin x$ and $\cos x$. This is represented by a *finite* Fourier series with as many terms as required by the degree of the polynomial, so that all higher Fourier coefficients are equal to zero. Another example is given by the elliptic ϑ series, which we shall meet in a heat conduction problem in §15; its Fourier coefficients C_k decrease as fast as $e^{-\alpha k^2}$.

It further follows from our theorem that the sum ΣA_k^2 which appears in the relation of completeness converges like Σk^{-2} for every function $f(x)$ which has a finite number of jumps and which is differentiable everywhere else (case $n = 0$ of our theorem). An example of this is given by our function (2.1) where ΣA_k^2 converges, although ΣA_k diverges. This function also shows that the relation of completeness does not insure representability of the function at every point (this has already been noted on p. 5). Namely, if we sharpen definition (2.1) by putting $f = 1$ for $x \geq 0$ and $f = -1$ for $x < 0$, then f if not represented by the Fourier series (2.2) at the point $x = 0$, for there the series converges to 0.

A further illustration of our theorem is given by the sine and cosine series which were derived at the end of the last section. The expressions of the functions which are represented by these series were valid only for the interval $0 < x < \pi$. We complete these expressions by adjoining the corresponding expressions for the interval $-\pi < x < 0$. The latter are obtained simply from the remark that the cosine series are even functions of x, and the sine series are odd. The expressions thus obtained are written below inside the $\{\ \}$ to the right of the semicolon. We therefore complete the equations (2.9), (2.12), (2.15), (2.17) as follows:

$$(5)\quad \left\{\frac{\pi}{4};\ -\frac{\pi}{4}\right\} = \sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots$$

$$(6)\quad \left\{\frac{\pi}{4}\left(\frac{\pi}{2} - x\right);\quad \frac{\pi}{4}\left(\frac{\pi}{2} + x\right)\right\} = \cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \cdots$$

$$(7)\quad \left\{\frac{\pi}{8}(\pi x - x^2);\ \frac{\pi}{8}(\pi x + x^2)\right\} = \sin x + \frac{1}{3^3}\sin 3x + \frac{1}{5^3}\sin 5x + \cdots$$

$$(8)\quad \left\{\frac{\pi}{8}\left(\frac{\pi^3}{12} - \frac{\pi x^2}{2} + \frac{x^3}{3}\right);\ \frac{\pi}{8}\left(\frac{\pi^3}{12} - \frac{\pi x^2}{2} - \frac{x^3}{3}\right)\right\} = \cos x + \frac{1}{3^4}\cos 3x + \cdots.$$

Here the functions which are represented possess successively stronger continuity properties: in (5) the function possesses discontinuities at the points $x = 0$ and $x = \pm\pi$, in (6) the function is continuous but the first derivative is discontinuous, in (7) the function and its first derivative are continuous but the second derivative is discontinuous; in (8) the function and its first two derivatives are continuous but the third derivative is not. The discontinuity arising in each case is the same as that of the function in (5) and it appears at the same points $x = 0$ and $x = \pm\pi$ corresponding to the fact that each succeeding function was obtained from the previous one by integration.

Figure 5 illustrates this. Its curves 0,1,2,3 represent the left sides of (5),(6),(7),(8). The discontinuity of the tangent to the curve 1 at $x = 0$ strikes the eye; the discontinuity of the curvature of 2 at $x = 0$ can be deduced from the behavior of the two mirror image parabolas which meet there. Curve 3 consists of two cubic parabolas, that osculate with continuous curvature. The scale, which for convenience has been chosen differently for the different curves, can be seen by the ordinates of the maximal values which have been inserted on the right hand side.

The *increasing continuity* of our curves 0 to 3 has its counterpart in the *increasing rate of convergence* of the Fourier series on the right sides of eqs. (5) to (8): in (5) we have a decrease of the coefficients like $1/k$, in general, in accord with our theorem, we have a decrease with k^{-n-1}, where n is the order of the first discontinuous derivative of the represented function.

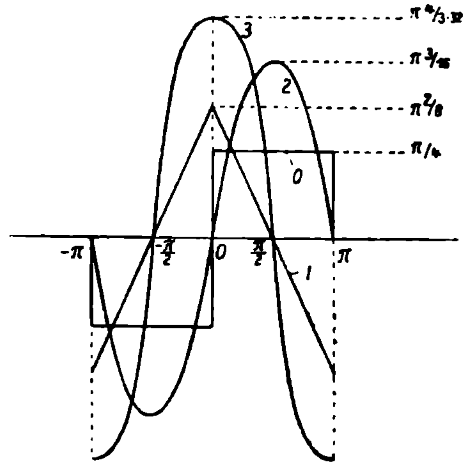


Fig. 5. Four curves 0,1,2,3, obtained by successive integration. Increasing continuity at $x = 0$: 0 discontinuous in the ordinate, 1 in the tangent, 2 in the curvature, 3 in the third derivative.

The convergence of Fourier series stands in a marked contrast to that of Taylor series. The former depends only on the continuity of the function to be represented and its derivatives on the real axis, the latter depends also on the position of the singularities in the complex domain. (Indeed the singular point nearest the origin of expansion in the complex plane determines the radius of convergence of the Taylor series.) Accordingly the principles of the two expansions are basically different: for Fourier series we have an *oscillating approach* over the entire range of the interval of representation, for Taylor series we have an *osculating approach* at its origin. We shall return to this in §6.

§ 4. Passage to the Fourier Integral

The interval of representation $-\pi < x < \pi$ can be changed in many ways. Not only can it be displaced, as remarked on p. 14, but also its length can be changed, e.g., to $-a < z < +a$ for arbitrary a. This is done by the substitution

$$x = \frac{\pi z}{a}, \tag{1}$$

which transforms (1.7) into

$$\left.\begin{matrix} A_k \\ B_k \end{matrix}\right\} = \frac{1}{a}\int_{-a}^{+a} f(z) \begin{matrix}\cos \\ \sin\end{matrix} \frac{\pi k z}{a}\, dz, \qquad A_0 = \frac{1}{2a}\int_{-a}^{+a} f(z)\, dz. \tag{2}$$

In the more convenient complex way of writing (1.12), one then has

$$f(z) = \sum_{-\infty}^{+\infty} C_k\, e^{i\frac{\pi}{a}kz}, \qquad C_k = \frac{1}{2a}\int_{-a}^{+a} f(\zeta)\, e^{-i\frac{\pi}{a}k\zeta}\, d\zeta. \tag{3}$$

We may obviously consider also the more general interval $b < z < c$ by substituting

$$(4) \qquad x = \alpha z + \beta, \qquad \alpha = \frac{2\pi}{c-b}, \qquad \beta = -\pi\frac{c+b}{c-b}$$

The formulas (2) then become

$$(5) \qquad \left.\begin{matrix} A_k \\ B_k \end{matrix}\right\} = \frac{2}{c-b}\int_b^c f(z) \begin{matrix}\cos\\ \sin\end{matrix} k(\alpha z + \beta)\,dz, \qquad A_0 = \frac{1}{c-b}\int_b^c f(z)\,dz.$$

In this connection we mention some "pure sine and cosine series" that appear in Fourier's work. One considers a function $f(x)$ which is given only in the interval $0 < x < \pi$ say, and which is to be continued to the negative side in an odd or even manner. For example, one gets for odd continuation

$$f(x) = \sum_{k=1}^{\infty} B_k \sin k x, \qquad B_k = \frac{2}{\pi}\int_0^{\pi} f(x) \sin k x\, dx,$$

See also exercise I.3.

Starting from (3) we take a to be very large. The sequence of values

$$\omega_k = \frac{\pi}{a} k$$

then becomes dense, for which reason we shall write ω instead of ω_k from now on. For the difference of two consecutive ω_k we write correspondingly

$$d\omega = \frac{\pi}{a}, \qquad \frac{1}{a} = \frac{d\omega}{\pi}.$$

If in (3) we replace the symbols z, ζ by the previous ones x, ξ then we obtain

$$(6) \qquad C_k = \frac{d\omega}{2\pi}\int_{-a}^{+a} f(\xi)\, e^{-i\omega\xi}\, d\xi.$$

For the moment we avoid calling the limits of this integral $-\infty$ and $+\infty$

Introducing (6) into the infinite series (3) for $f(x)$, replacing the summation by integration, and denoting the limits of integration for the time being by $\pm\Omega$, we get:

$$(7)\qquad f(x) = \lim_{\Omega\to\infty}\lim_{a\to\infty}\frac{1}{2\pi}\int_{-\Omega}^{+\Omega} e^{i\omega x}\,d\omega\int_{-a}^{+a} f(\xi)\,e^{-i\omega\xi}\,d\xi .$$

The order of passage to the limit indicated here is obviously necessary: if the passage to the limit $\Omega\to\infty$ were carried out first, we would obtain the completely meaningless integral

$$\int_{-\infty}^{+\infty} e^{i\omega(x-\xi)}\,d\omega$$

On the other hand $f(\xi)$ must vanish for $\xi\to\pm\infty$ in order that the first limit for $a\to\infty$ have a meaning. We do not have to investigate how fast $f\to 0$ in order that the other passage to the limit be possible, since for all suitably formulated physical problems this convergence to 0 will be "sufficiently rapid."

After this preliminary discussion we shall further abbreviate the more exact form of (7) by writing:

$$(8)\qquad f(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} d\omega\int_{-\infty}^{+\infty} f(\xi)\,e^{i\omega(x-\xi)}\,d\xi .$$

From this we pass to the *real* form of the Fourier integral (8) as it is commonly given in the literature. We set

$$e^{i\omega(x-\xi)} = \cos\omega\,(x-\xi) + i\sin\omega\,(x-\xi) .$$

Here the sine is an odd function of ω, and hence vanishes on integration from $-\infty$ to $+\infty$; the cosine, being even in ω, yields twice the integral taken from 0 to ∞. We therefore have

$$(9)\qquad f(x) = \frac{1}{\pi}\int_{0}^{+\infty} d\omega\int_{-\infty}^{+\infty} f(\xi)\cos\omega\,(x-\xi)\,d\xi ,$$

by which we do not wish to imply that the real form is better or simpler than our complex form (8). We can write instead of (9):

$$(10)\qquad f(x) = \int_0^{\infty} a(\omega)\cos\omega x\,d\omega + \int_0^{\infty} b(\omega)\sin\omega x\,d\omega$$

where

$$(10\,a)\quad a(\omega)=\frac{1}{\pi}\int_{-\infty}^{+\infty} f(\xi)\cos\omega\,\xi\,d\xi, \qquad b(\omega)=\frac{1}{\pi}\int_{-\infty}^{+\infty} f(\xi)\sin\omega\,\xi\,d\xi.$$

In particular $b(\omega)$ must vanish if $f(x)$ is even, $a(\omega)$ if $f(x)$ is odd. We then have corresponding to the above "pure cosine or sine series," a "pure cosine or sine integral." One or the other can be produced whenever $f(x)$ is given only for $x > 0$, by continuing $f(x)$ as an even or odd function to the negative side. We then write explicitly:
for even continuation

$$(11\,a)\quad f(x)=\int_0^\infty a(\omega)\cos\omega\,x\,d\omega, \qquad a(\omega)=\frac{2}{\pi}\int_0^\infty f(\xi)\cos\omega\,\xi\,d\xi,$$

for odd continuation

$$(11\,b)\quad f(x)=\int_0^\infty b(\omega)\sin\omega\,x\,d\omega, \qquad b(\omega)=\frac{2}{\pi}\int_0^\infty f(\xi)\sin\omega\,\xi\,d\xi.$$

The usefulness of this procedure will become apparent to us in some particular problems of heat conduction below.

We denoted the variable of integration by ω deliberately. In general one denotes the *frequency* in oscillation processes by ω. Let us therefore, for the time being, think of x as the *time coordinate;* then in equation (10) we have the *decomposition of an arbitrary process in time,* $f(x)$, *into its harmonic components.* In the *Fourier integral* one is concerned with a *continuous spectrum,* which ranges over all frequencies from $\omega = 0$ to $\omega = \infty$ in the *Fourier series* with a *discrete spectrum,* consisting of a fundamental tone plus harmonic overtones. Here the following fact must be kept in mind: when a physicist determines the spectrum of a process with a suitable spectral apparatus, he finds only the *amplitude* belonging to the frequency ω, while the phase of the partial oscillations remains unknown to him. In our notation the amplitude corresponds to the quantity

$$c(\omega)=\sqrt{a^2(\omega)+b^2(\omega)},$$

the phase, $\gamma(\omega)$, is given by the ratio b/a. The relation between these various quantities is best given as

$$(12)\qquad c(\omega)\,e^{i\gamma(\omega)}=a(\omega)+i\,b(\omega).$$

The Fourier integral which describes the process completely uses both quantities a and b, i.e., *both* amplitude *and* phase. The observable spectrum therefore yields, so to speak, only half the information which is contained in the Fourier integral.

This is noted markedly in the "Fourier analysis of crystals," which is so successfully carried out nowadays. Here only the *intensities* of the crystal reflexes, i.e., the squares of the *amplitudes*, can be observed; for a complete knowledge of the crystal structure one would have to know the *phases* too. This defect can only be partially removed by symmetry considerations.

In exercise I.4 we shall deal with the spectra of diverse oscillation processes as examples for the theory of the Fourier integral and at the same time as completion of the spectral theory.

Once more we return to the complex form of the Fourier integral and split it into two parts

$$(13) \quad f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \varphi(\omega)\, e^{i\omega x}\, d\omega, \qquad \varphi(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x)\, e^{-ix\omega}\, dx.$$

which together are equivalent to (8). Disregarding the splitting of the denominator 2π into $\sqrt{2\pi} \cdot \sqrt{2\pi}$, which was done mainly for reasons of symmetry, and disregarding the notation of the variable of integration in the second equation, we have $\varphi(\omega)$ identical with the quantity $a(\omega) - i\,b(\omega)$ defined in (10a); it therefore contains information concerning both the amplitude and the phase of the oscillating process $f(x)$.

Moreover (10) shows that the two functions f and φ have a *reciprocal* relation: one is determined by the other, whether we regard f as known and φ as unknown or conversely, and the determination in each case is by "integral equations" of exactly the same character. One says that one function is the *Fourier transform* of the other. In (13) we have a particularly elegant formulation of Fourier's integral theorem.

So far we have spoken only of functions $f(x)$ of *one* variable. It is obvious that a function of several variables can be developed into a Fourier series or integral with respect to any one of the variables. By developing with respect to x,y,z for example we obtain a triply infinite Fourier series and sixfold Fourier integrals. We do not wish to write here the somewhat lengthy formulas since we shall have ample opportunity to explain them in their applications.

§ 5. Development by Spherical Harmonics

We do not claim that the path we shall pursue is the most convenient approach to the theory of spherical harmonics; but it proceeds immediately from the discussion of §1, needs no preparation from the theory of differential equations, and leads to interesting points of view on far reaching generalizations.

We consider the problem: Approximate a function $f(x)$ given in the interval $-1 < x < +1$ by a sequence of polynomials $P_0, P_1, P_2, \ldots P_k, \ldots P_n$ of degrees $0, 1, 2, \ldots k, \ldots n$ in the *manner which is the best possible from the point of view of the method of least squares.* We form an n-th approximation of the form

$$S_n = \sum_{k=0}^{n} A_k P_k \tag{1}$$

and reduce the mean error

$$M = \frac{1}{2} \int_{-1}^{+1} [f(x) - S_n]^2 \, dx \tag{2}$$

to a minimum through choice of A_k, just as in (1.3). This leads to the $n + 1$ equations:

$$\int_{-1}^{+1} [f(x) - S_n] \, P_k \, dx = 0, \quad k = 0, 1, \ldots, n. \tag{3}$$

just as in (1.4). This minimal requirement we complete by a requirement concerning the amount of calculation that will be needed: the coefficients A_k which are to be calculated from (3) in the n-th approximation, shall also be valid in the $(n + 1)$-st and in all subsequent approximations; they shall represent the *final* A_k for all $k \leq n$, and the finer approximations are to complete their determination by yielding the A_k for $k > n$. In §1, p. 4 this finality of the A_k resulted from the known orthogonality of the trigonometric functions. Here, conversely, the *requirement of finality* will be seen to imply the *orthogonality* of the P_k.

The proof is very simple. Equation (3), written explicitly, reads (we omit in the following the limits of integration ± 1):

$$A_0 \int P_0 P_k \, dx + A_1 \int P_1 P_k \, dx + \cdots + A_n \int P_n P_k \, dx = \int f(x) \, P_k \, dx. \tag{4}$$

Since the right side is independent of n and the A_i are to be final, this equation retains its validity for the $(n + 1)$-st approximation S_{n+1}, except that on the left side we add the term

$$A_{n+1} \int P_{n+1} P_k \, dx$$

Equation (4) implies that this term must vanish, and since A_{n+1} does not vanish (except for special choice of $f(x)$), the integral must vanish for all k for which (4) is valid, i.e., for all $k \leqq n$. But this implies that P_{n+1} is *orthogonal* to $P_0, P_1, \ldots P_n$ for arbitrary n. Hence, if we take P_0 and P_1 orthogonal to each other, our requirement of finality implies the *general condition of orthogonality*

(5) $$\int P_n P_m \, dx = 0, \qquad m \neq n.$$

Using (5) we obtain from (4)

(6) $$A_k \int P_k^2 \, dx = \int f(x) \, P_k(x) \, dx .$$

The A_k are therefore determined individually if we add a convention about the *normalizing integral* on the left side of (6). The most obvious procedure would be to set it directly equal to 1, and indeed we shall do this in the general theory of characteristic functions. Here we prefer to follow historical usage and require instead that

(7) $$P_n(1) = 1 .$$

This normalizing condition has an advantage in that, as we shall see, all the coefficients in P_n become rational numbers.

We now pass to the recursive calculation of $P_0, P_1, P_2, \ldots$ from (5) and (7). P_0 is a constant, which according to (7), must be set equal to 1. In the linear function $P_1 = a\,x + b$ we see from (5), after setting $n = 0$ and $m = 1$, that $b = 0$ and from (7) that $a = 1$. After setting $P_2 = a\,x^2 + b\,x + c$ we obtain

$$\int P_2 P_0 \, dx = \frac{2}{3} a + 2c = 0 ; \quad \text{hence} \quad c = -\frac{a}{3} ;$$

$$\int P_2 P_1 \, dx = \frac{2}{3} b = 0 ; \qquad \text{hence} \quad b = 0 ;$$

Therefore $P_2 = a\,(x^2 - \frac{1}{3})$ and by (7)

$$a = \frac{3}{2}, \qquad P_2 = \frac{3}{2} x^2 - \frac{1}{2} .$$

Correspondingly we find

$$P_3 = \frac{5}{2} x^3 - \frac{3}{2} x , \qquad P_4 = \frac{35}{8} x^4 - \frac{15}{4} x^2 + \frac{3}{8} \text{, etc.}$$

The P_n are therefore completely determined by our two requirements, the P_{2n} as even, the P_{2n+1} as odd polynomials with rational coefficients.

More transparent than the recursive process is the following explicit representation:

$$P_n(x) = \frac{1}{2^n\, n!} \frac{d^n}{dx^n} (x^2 - 1)^n . \tag{8}$$

We see that $P_n(x)$ as defined by (8) satisfies condition (7) as follows: for $x \to 1$ we have to carry out the n-fold differentiation solely for the factor $(x - 1)^n$, whereby we obtain $n!$; the factor $(x + 1)^n$ becomes equal to 2^n; equation (8) therefore does imply that $P_n(1) = 1$.

It remains to be proven that (8) satisfies the orthogonality condition (5), which is equivalent to our "condition of finality." To this end we introduce the notation

$$D_{k,l} = \frac{d^k}{dx^k}(x^2 - 1)^l \tag{9}$$

and write the left side of (5) (suppressing the constant factor which is immaterial here) as

$$\int_{-1}^{+1} D_{nn}\, D_{mm}\, dx ,$$

where we take, say, $m > n$. We now reduce the order of differentiation of the second factot D_{mm} by integration by parts; this increases the order of differentiation of D_{nn}. The terms which fall outside the integral sign will vanish for $x = \pm 1$, since in $D_{m-1,m}$ according to (9) one factor $x^2 - 1$ remains. Repeating this process we get

$$\begin{aligned} &\int D_{n,n} \cdot D_{m,m}\, dx = -\int D_{n+1,n} \cdot D_{m-1,m}\, dx = \\ &\int D_{n+2,n} \cdot D_{m-2,m}\, dx = \cdots = (-1)^n \int D_{2n,n} \cdot D_{m-n,m}\, dx . \end{aligned} \tag{10}$$

Here according to (9) $D_{2n,n}$ is a constant, namely $(2n)!$ Hence

$$\begin{aligned} \int D_{n,n} \cdot D_{m,m}\, dx &= (-1)^n (2n)! \int D_{m-n,m}\, dx \\ &= (-1)^n (2n)!\, D_{m-n-1,m} \Big|_{-1}^{+1} . \end{aligned} \tag{11}$$

This vanishes, since the number $m - n - 1$ of differentiations that still remain to be carried out is less than the number m of factors $x - 1$ and $x + 1$ which are to be differentiated. This deduction is valid for $m = n + 1$, too, and fails only for $m = n$. The orthogonality is therefore proved for all $m \neq n$.

At the same time the method just used provides a way of calculating the normalizing integral of (6):

$$\int P_k^2\,dx = \left(\frac{1}{2^k k!}\right)^2 \int D_{k,k}\cdot D_{k,k}\,dx.$$

Using the first line of (11) for $m = n = k$, we obtain

$$\int P_k^2\,dx = \frac{(-1)^k(2k)!}{(2^k k!)^2}\int D_{0,k}\,dx = \frac{(2k)!}{(2\cdot 4\cdot 6\ldots 2k)^2}\int (1-x^2)^k\,dx.$$

The numerical factor in front of the last integral is

$$z = \frac{1\cdot 3\cdot 5\cdot \ldots \cdot(2k-1)}{2\cdot 4\cdot 6\cdot \ldots \cdot 2k};$$

under the substitution $x = \cos\vartheta$ the integral itself goes over into the well known form

$$\int_0^\pi \sin^{2k+1}\vartheta\,d\vartheta = 2\cdot\frac{2\cdot 4\cdot 6\ldots 2k}{3\cdot 5\cdot 7\ldots(2k+1)} = \frac{2}{2k+1}\cdot\frac{1}{z}.$$

Therefore, one obtains

$$\int P_k^2\,dx = \frac{2}{2k+1} = \frac{1}{k+\tfrac{1}{2}}. \tag{12}$$

Equation (6) then gives

$$A_k = (k+{}^1/_2)\int f(x)\,P_k(x)\,dx. \tag{13}$$

Substituting this in equation (1) of the n-th approximation S_n and letting $n\to\infty$ we get (assuming convergence and the completeness of the system of functions P):

$$f(x) = \sum_{k=0}^{\infty}(k+{}^1/_2)\int_{-1}^{+1} f(\xi)\,P_k(\xi)\,d\xi\cdot P_k(x). \tag{14}$$

The two assumptions just mentioned can be justified here, just as in the case of Fourier series, by consideration of the limiting value of the mean square error. The k-th approximating function has k zeros in the interval of approximation just as before, except that now they are not equally spaced. The approach to the given function, f, proceeds, here too, through more and more frequent *oscillations*. Also, we find Gibbs' phenomenon at the points of discontinuity, etc.

§ 6. Generalizations: Oscillating and Osculating Approximations. Anharmonic Fourier Analysis. An Example of Non-Final Determination of Coefficients

The following question suggests itself: Why are the two series different, despite the identical nature of the approximation processes? Since we saw that the form of the $P_n(x)$ was completely determined by our approximation requirements, we might think, e.g., that the pure cosine series (expansion of an even function) would go over into a series of spherical harmonics, if in the former we set $\cos\varphi = x$, because then $\cos k\varphi$ becomes a polynomial of degree k in x just like $P_k(x)$, and the interval of expansion $0 < \varphi < \pi$ becomes the interval $+1 > x > -1$. *But the individual infinitesimal elements of this interval receive a different weight g in each case* since

$$d\varphi = -\frac{dx}{\sqrt{1-x^2}}.$$

Whereas in the Fourier approximation we associate the same weight with all $d\varphi$, the endpoints $x = \pm 1$ of the interval in the x scale seem to be favored since $g(x) = 1/\sqrt{1-x^2}$. At these points the function is better approximated than at the middle of the interval. The opposite is obviously the case for approximations by spherical harmonics which, translated to the φ-scale, discriminate against the endpoints of the interval since $g(\varphi) = \sin\varphi$. Pictorially speaking, in the case of Fourier series, one deals with a uniformly weighted unit semicircle between $\varphi = 0$ and π, which, under orthogonal projection on the diameter between $x = -1$ and $+1$, yields a non-uniform density; on the other hand the case of spherical harmonics deals with a uniformly weighted diameter, which corresponds to a non-uniformly weighted semicircle.

A. Oscillating and Osculating Approximation

These different *distributions of weight g* (that is, densities) are the factors that, in conjunction with the delimitation of the interval of expansion, distinguish among the different series expansions common in mathematical physics. Here we only mention the expansions in Hermite- and Laguerre-polynomials because of their importance for wave mechanics. We shall not concern ourselves here with their formal representation — they can be obtained from the requirement of a best possible calculation of the coefficients satisfying a condition of finality, just as in the case of spherical harmonics. (See exercise I.6, where the usual normalizations are given; orthogonality would again be the necessary result of these requirements.) We restrict ourselves here to a tabulation of the most important characteristics of both polynomial series:

	HERMITE	LAGUERRE
Interval	$-\infty < x < +\infty$	$0 < x < \infty$
Weight $g(x)$	e^{-x^2}	e^{-x}
Orthogonality condition for $m \neq n$	$\int_{-\infty}^{+\infty} H_n H_m e^{-x^2} dx = 0$	$\int_0^{\infty} L_n L_m e^{-x} dx = 0$

For these series, just as for Fourier series and spherical harmonics, the approach to the given function, f, is through closer and closer *oscillations*. However, from the calculus we know a series whose character is *osculating* rather than *oscillating*, namely the *Taylor series*. In the case of Taylor series the consecutive approximations S_n osculate the curve to be represented in such a way that at a given point S_n has the same derivatives as f up to and including $f^{(n)}$ The graphic representation of the power series of sin x (Fig. 6) demonstrates this without further explanation.

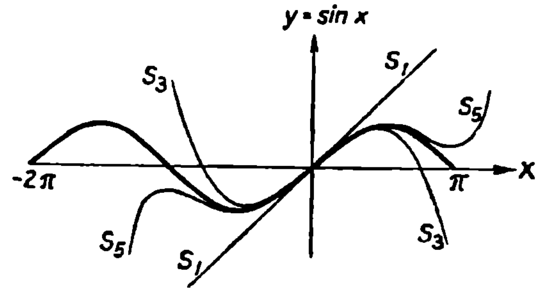


Fig. 6. The Taylor expansion of sin x (heavy line) and its approximations

$$S_1 = x,\ S_3 = x - \frac{x^3}{3!},$$
$$S_5 = x - \frac{x^3}{3!} + \frac{x^5}{5!}.$$

Here the total accuracy is concentrated at a single point. Following Dirac we can express this succinctly as follows: $g(x)$ has degenerated into a δ function. Dirac defines, as an analogue to the algebraic symbol δ_{kl} of (1.6), a highly discontinuous function $\delta(x \mid x_0)$

(1) $$\delta(x \mid x_0) = \begin{cases} 0 & x \neq x_0, \\ \infty & x = x_0, \end{cases} \qquad \int_{x_0-\varepsilon}^{x_0+\varepsilon} \delta(x \mid x_0)\, dx = 1$$

for arbitrary ε For the Taylor series of Fig. 6, where x_0 has been set equal to 0, we get

(1a) $$g(x) = \delta(x \mid 0).$$

B. ANHARMONIC FOURIER ANALYSIS

Whereas in §1–3 we considered only Fourier series which proceed according to *harmonic* (integral) overtones of a fundamental tone, we now consider the problem of expanding an arbitrary function $f(x)$ in the interval $0 < x < \pi$ into a series of the form

(2) $$f(x) = B_1 \sin \lambda_1 x + B_2 \sin \lambda_2 x + B_3 \sin \lambda_3 x + \cdots$$

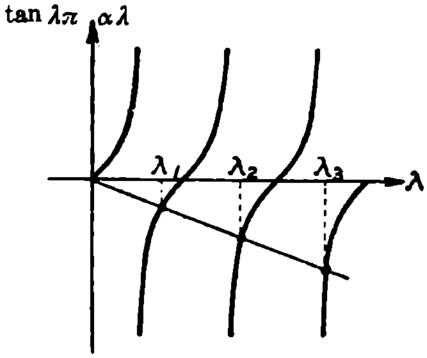


Fig. 7. Diagram of the transcendental equation $\tan \lambda\pi = \alpha\lambda$ $\alpha < 0$. In the ordinate both $y = \tan \lambda\pi$ and $y = \alpha\lambda$ have been drawn. The intersections yield the roots, λ_k, of the equation. $\lambda_0 = 0$ is not to be considered as a root; for $n \to \infty$ we get asymptotically $\lambda_n = n - \frac{1}{2}$.

where the λ_k are given as the roots of a transcendental equation, e.g.,

(2a) $$\tan \lambda \pi = \alpha \lambda$$

(α being an arbitrary number). We do this for use in problems of heat conduction (see §16). The fact that (2a) has infinitely many roots is seen directly from Fig. 7 where λ has been drawn as the abscissa and both $\tan \lambda\pi$ and $\alpha \lambda$ as ordinates. We shall meet another equation of character similar to (2a) in exercise II.1.

We first show that the functions $\sin \lambda_k x$ form an orthogonal system with weighting factor $g(x) = 1$, i.e., that

(3) $$\int_0^\pi \sin \lambda_k x \sin \lambda_l x \, dx = 0 \qquad k \neq l.$$

In fact, by passing from the product of sines to the cosines of the sums and differenees, we obtain for the left hand of (3)

$$\frac{\lambda_k \lambda_l}{\lambda_k^2 - \lambda_l^2} \cos \lambda_k \pi \, \cos \lambda_l \pi \left(\frac{\tan \lambda_k \pi}{\lambda_k} - \frac{\tan \lambda_l \pi}{\lambda_l} \right),$$

where the expression inside the brackets now vanishes because of (2a). In the same manner we find for $k = l$

(3a) $$\int_0^\pi \sin^2 \lambda_k x \, dx = \frac{\pi}{2}\left(1 - \frac{1}{\lambda_k \pi} \sin \lambda_k \pi \cdot \cos \lambda_k \pi\right).$$

This calculation of (3) and (3a) which is based on special trigonometric identities, will receive a less formal treatment in §16 where it will be reduced to an application of Green's theorem.

From (3) and (3a) one obtains the following value for the expansion coefficients B_k in (1):

$$B_k = \frac{2}{\pi} \int_0^{\pi} \frac{f(x) \sin \lambda_k x}{1 - \frac{\sin 2\lambda_k \pi}{2\lambda_k \pi}} dx. \tag{3b}$$

This value for B_k is *final* in the sense of p. 22, since it is independent of n and minimizes the mean square error of the approximation

$$S_n = \sum_{k=1}^{n} B_k \sin \lambda_k x$$

At the same time this settles the question of convergence and completeness, if for $n \to \infty$ the mean square error approaches zero.

C. An Example of a Non-Final Determination of Coefficients

As preparation for an optical (or rather "quasi-optical") application, we shall consider a much more involved case in which the requirement of finality is *not* satisfied. Let us consider a metal mirror in the shape of a circular cylinder (see Fig. 8). The electric vector of the total oscillation, which we take as perpendicular to the plane of the drawing, is composed of the incoming wave, represented on the mirror by

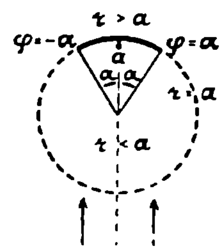


Fig. 8. Reflection of an incoming "quasioptical" wave on a circular cylinder mirror of opening $\varphi = \alpha$ and radius $r = a$.

$$w = -f(\varphi), \quad -\alpha < \varphi < +\alpha, \quad r = a \tag{4}$$

and of the reflected (refracted, scattered) wave. Let the latter be represented by:

$$u = u(r, \varphi), \quad -\pi < \varphi < +\pi, \quad r < a \text{, inner field,}$$

$$v = v(r, \varphi), \quad -\pi < \varphi < +\pi, \quad r > a \text{, outer field.}$$

We then have to demand

$$u + w = v + w = 0 \quad \text{for } r = a \text{ and } |\varphi| < \alpha, \tag{5}$$

$$u = v, \quad \frac{\partial u}{\partial r} = \frac{\partial v}{\partial r} \qquad \text{for } r = a \text{ and } |\varphi| > \alpha, \tag{6}$$

the former on account of the assumed infinite conductivity of the metal mirror, the latter on account of the required continuous passage from the inner to the outer field.

Assuming w to be symmetric with respect to the axis of the mirror (as, for example in the case of a plane wave proceeding in that direction), we write[7]

$$\begin{aligned} u &= \sum_n C_n\, g_n(r) \cos n\varphi, \\ v &= \sum_n D_n\, h_n(r) \cos n\varphi. \end{aligned} \tag{7}$$

g_n and h_n will turn out to be Bessel and Hankel functions, respectively (see §19); they can be chosen so that

$$g_n(a) = h_n(a) = 1$$

Equation (5) and the first equation (6) then imply

$$\sum_n C_n \cos n\varphi = \sum_n D_n \cos n\varphi = f(\varphi) \qquad |\varphi| < \alpha \tag{8}$$

and

$$\sum_n C_n \cos n\varphi = \sum_n D_n \cos n\varphi \qquad |\varphi| > \alpha. \tag{9}$$

respectively. From these two equations it follows that

$$\sum_n (C_n - D_n) \cos n\varphi = 0 \quad \text{for all } \varphi,$$

hence, whether the preceding summations are extended over all integers n or only over the first N integers (the more general case), we have

$$D_n = C_n.$$

This satisfies (9) while (8) still requires

$$\sum_n C_n \cos n\varphi = f(\varphi) \text{ for } |\varphi| < \alpha. \tag{10}$$

In addition to this we have to satisfy the second equation (6) which on account of (7) reads:

[7] In view of the notations to be used in Chapter IV, it is advisable here to change the index of summation from k to n. For the previous n we shall write N, and instead of l we shall use m.

$$\sum_n C_n \gamma_n \cos n\varphi = 0 \qquad \text{for } |\varphi| > \alpha \tag{11}$$

$$\gamma_n = a\left(\frac{dg_n(r)}{dr} - \frac{dh_n(r)}{dr}\right)_{r=a} \tag{11a}$$

We add the factor a before the parentheses here, as we may by (11), in order to make γ_n a pure number. Equations (10) and (11) together determine the C_n.

Here the way is again shown by the method of least squares. We consider the square errors corresponding to the equations (10) and (11)

$$\int_0^\alpha \left(f(\varphi) - \sum_{n=0}^N C_n \cos n\varphi\right)^2 d\varphi \quad \text{and} \quad \int_\alpha^\pi \left(\sum_{n=0}^N C_n \gamma_n \cos n\varphi\right)^2 d\varphi.$$

The sum of these two is to be minimized through choice of the C_n. By differentiation with respect to the C_n this yields a system of $N + 1$ linear equations for $C_0, \ldots C_n, \ldots C_N$, of which the $(m + 1)$-st equation is:

$$\sum_{n=0}^N C_n \left\{\int_0^\alpha \cos n\varphi \cos m\varphi \, d\varphi + \gamma_n \gamma_m \int_\alpha^\pi \cos n\varphi \cos m\varphi \, d\varphi\right\} = \int_0^\alpha f(\varphi) \cos m\varphi \, d\varphi. \tag{12}$$

If we pass to the limit $N \to \infty$ we obtain an *infinite system of linear equations for the infinitely many unknowns* C_n, which are in general of no interest to us. We must postpone further treatment of this problem until appendix I of Chapter IV, for only there shall we have the necessary values of the parameters γ_n. The corresponding spatial problem, where we have a spherical segment instead of a circular cylinder segment, would lead in the limit $N \to \infty$ to an infinite system of linear equations, in which $P_n(\cos\vartheta)$ would replace $\cos n\varphi$ (by ϑ we denote here the angle measured from the axis of symmetry of the spherical mirror). This problem too will be treated in appendix I of Chapter IV. At present we call attention only to the difference in method between those problems in which the method of least squares leads to a definitive calculation of the *individual coefficients* C, and those problems in which the "requirement of finality" is not satisfied and in which therefore, *the totality of the* C_n *must be determined from the totality of minimality conditions.*

CHAPTER II

Introduction to Partial Differential Equations

§ 7. How the Simplest Partial Differential Equations Arise

The potential equation

(1) $$\Delta u = 0 \quad \text{or} \quad (1\text{a}) \quad \Delta u = -(4\pi)\varrho$$

is known in the *theory of gravitation* as the expression of the field-action approach, as opposed to the action-at-a-distance approach of Newton. The *Laplace operator* is defined as

(2) $$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \text{div grad}.$$

The same equations (1) and (1a) are fundamental for *eloctrostatic* and *magnetic fields*, (1) in empty space, (1a) in the presence of a source of density ϱ the factor 4π in (1a) has been put in parentheses since it can be removed by a proper choice of units.

Equation (1) appears also in the *hydrodynamics* of incompressible and irrotational fluids, u standing for the velocity potential. We also mention the two-dimensional potential equation

(3) $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

as the basis of Riemannian function theory, which we may characterize as the "field theory" of the analytic functions $f(x + iy)$.

Equally well known is the wave equation

(4) $$\Delta u = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2}.$$

It is fundamental in acoustics (c = velocity of sound). It is also fundamental in the electrodynamics of variable fields (c = velocity of light), and therefore in optics. In the special theory of relativity one may write (4) as the four-dimensional potential equation

(5) $$\Box\, u = 0 \qquad \text{with} \qquad \Box = \sum_{k=1}^{4} \frac{\partial^2}{\partial x_k^2}$$

by introducing the fourth coordinate x_4 (or x_0) $= ict$ in addition to the three spatial coordinates x_1, x_2, x_3. For an oscillating membrane we have (4) with two spatial dimensions, for an oscillating string we have one spatial dimension. In the latter case we write

(6) $$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \qquad \text{or sometimes} \qquad (6\text{a}) \quad \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0,$$

setting, for the time being, $y = ct$ (not $y = ict$). Neither membrane nor string has a proper elasticity; the constant c is computed from the tension imposed from outside and from the density per unit of area or of length.

In the general theory of elasticity one has, as a special case, the differential equation for the transverse vibrations of a thin disc

(7) $$\Delta\Delta u = -\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}, \qquad \Delta\Delta = \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4};$$

for reasons of dimensionality c here does not stand for the velocity of sound in the elastic material, as it does in acoustics, but is computed from the elasticity, density, and thickness of the disc. Analogously, the differential equation of an oscillating elastic rod is

(8) $$\frac{\partial^4 u}{\partial x^4} = -\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}.$$

This will be derived in exercise II.1, where the resulting characteristic frequencies will be compared with the acoustic frequencies of open and of covered pipes.

As a third type we add to the differential equations of states of equilibrium ((1) to (3)), and of oscillating processes ((4) to (8)), those of *equalization processes.* As their chief representative we shall here consider *heat conduction* (equalization of energy differences). We remark, however, that *diffusion* (equalization of differences of material densities), *fluid friction* (equalization of impulse differences), and pure *electric conduction* (equalization of differences of potential), follow the same pattern.

Let G be a vector of the magnitude and direction of the heat flow and let the initial point P be surrounded by an element of volume $d\tau$. Then div G $d\tau$ is the outflow of heat energy from $d\tau$ per unit of time. A decrease per unit of time in the amount of heat in $d\tau$, which we shall denote by $-\partial Q/\partial t$, corresponds to this. We then have

$$\text{div}\,\mathrm{G}\,d\tau = -\frac{\partial Q}{\partial t}. \tag{9}$$

Our heat conductor is here considered to be a rigid body so that we can neglect expansion; heat content is then the same as energy content. Now every increase dQ in heat causes an increase in the temperature of $d\tau$, every decrease $-dQ$ in heat causes a decrease in temperature. Denoting the temperature by u, we have

$$dQ = c\,dm\,du, \qquad dm = \varrho\,d\tau. \tag{10}$$

c being the specific heat (for a rigid body we need not distinguish between c_v and c_p). The factor dm is due to the fact that c is related to the unit of mass.

From (9) and (10) we get

$$\text{div}\,\mathrm{G} = -c\,\varrho\,\frac{\partial u}{\partial t}. \tag{11}$$

We now apply *Fourier's law*, which determines the relation between G and u. It states that for an isotropic medium

$$\mathrm{G} = -\varkappa\,\text{grad}\,u: \tag{12}$$

the flow of heat is in the direction of decreasing temperature and is proportional to the rate of this decrease. The factor of proportionality $\varkappa$ is called the *heat conductivity.*

Introducing (12) in (11) we get the differential equation of *heat conduction*

$$\Delta u = \frac{1}{k}\frac{\partial u}{\partial t}, \qquad k = \frac{\varkappa}{c\,\varrho}. \tag{13}$$

k is called the *temperature conductivity.*

Fourier's law was adapted to the case of *diffusion* by the physiologist Fick. Here u stands for the concentration of dissolved matter in the solvent, G for the *material flow* of the dissolved matter, and k for the *diffusion coefficient.* In the case of *inner friction* of an incompressible fluid, k stands for the *kinematic viscosity,* and (13) is the Navier-Stokes equation for laminar flow (i.e., flow in a fixed direction). Owing to the tensor character of this process equation (12) has no general validity here. The analogue of Fourier's law in the *electric* case is Ohm's law. Here u stands for the *potential,* G for the *specific electric current* (the current per unit of area of the conductor), and k for the *specific resistance of the* *conductor.* Equation (13) is of the type of Maxwell's equations in the case of pure Ohm conduction.

Schrödinger's equation of wave mechanics belongs formally to the same scheme, in particular in the force-free case, to which we restrict ourselves here:

(14) $$\Delta u = \frac{2\,m}{i\,\hbar}\,\frac{\partial u}{\partial t} \left\{ \begin{array}{l} \hbar = \text{Planck's constant divided by } 2\pi \\ m = \text{mass of the particle.} \end{array} \right.$$

However, owing to the fact that the real constant, k, of (13) is replaced here by the imaginary constant $i\,\hbar/2\,m$, equation (14) describes an oscillation rather than an equalization process. We see this in the passage to the case of periodicity in time, if we set

(14 a) $$u = \psi\, e^{-i\omega t}, \qquad \omega = \frac{W}{\hbar}, \qquad W = \text{energy of the state.}$$

Then (14) becomes

(15) $$\Delta\psi + C\,\psi = 0, \quad C = \frac{2\,m}{\hbar^2}\,W.$$

This is the same form as we would obtain from the wave equation (4) if we set $u = \psi \cdot \exp(-i\omega t)$ and let $C = \omega^2/c^2$

The so-called case of *linear heat conduction*, with the thermal state depending on only one variable x, will be treated in detail in the following chapter. In order to compare its differential equation with (3) and (6a), we write it in the form:

(16) $$\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial y} = 0, \quad y = k\,t.$$

Looking back on this sketchy survey one notices a family resemblance among the differential equations of physics. This stems from the *invariance under rotation and translation*, which must be demanded for the case of isotropic and homogeneous media. The differential operator of second order implied by this invariance is just the Laplace Δ. In the case of space-time invariance of relativity this is replaced by the corresponding four-dimensional $\Box$ of (15). For the case of an *anisotropic medium*, Δ must be replaced by a sum of all second derivatives with factors determined from the crystal constants. For the case of an *inhomogeneous medium* these factors will also be functions of position. We shall deal with such generalized differential expressions in the beginning of the next section.

The fact that we are dealing throughout with *partial differential* *equations* is due to the *field-action approach,* which is the basis of present day physics, according to which only neighboring elements of space can influence each other.

§ 8. Elliptic, Hyperbolic and Parabolic Type. Theory of Characteristics

We restrict ourselves to the case of two independent variables, x and y. The most general form of a *linear partial differential equation of second order* is then:

$$(1)\qquad L(u) \equiv A\,\frac{\partial^2 u}{\partial x^2} + 2\,B\,\frac{\partial^2 u}{\partial x\,\partial y} + C\,\frac{\partial^2 u}{\partial y^2} + D\,\frac{\partial u}{\partial x} + E\,\frac{\partial u}{\partial y} + F\,u = 0.$$

$A,B,\ldots,F$ being given functions of x and y having sufficiently many derivatives. For the present we may even consider the far more general equation:

$$(2)\qquad A\,\frac{\partial^2 u}{\partial x^2} + 2\,B\,\frac{\partial^2 u}{\partial x\,\partial y} + C\,\frac{\partial^2 u}{\partial y^2} = \Phi\left(u,\,\frac{\partial u}{\partial x},\,\frac{\partial u}{\partial y},\,x,\,y\right),$$

where Φ need not be linear in u, $\partial u/\partial x$, $\partial u/\partial y$.

We now investigate the conditions for the solvability of the following problem, which is put first in the mathematical theory of partial differential equations, although in the physical applications it is of secondary importance compared to certain boundary value problems considered later.

Let Γ be a given curve in the xy-plane along which both u and the derivative $\partial u/\partial n$ of u in the direction of the normal are prescribed. Does a solution of (2) that satisfies these initial conditions exist?

Preliminary remark: If u is given on Γ then so is $\partial u/\partial s$; but from $\partial u/\partial s$ and $\partial u/\partial n$ one can calculate $\partial u/\partial x$ and $\partial u/\partial y$. Therefore both u and its first derivatives are known on Γ.

We introduce the following abbreviations, which are common in the theory of surfaces:

$$p = \frac{\partial u}{\partial x},\quad q = \frac{\partial u}{\partial y},$$
$$r = \frac{\partial^2 u}{\partial x^2},\quad s = \frac{\partial^2 u}{\partial x\,\partial y},\quad t = \frac{\partial^2 u}{\partial y^2}.$$

Written in terms of r,s,t equation (2) reads:

$$(3)\qquad A\,r + 2\,B\,s + C\,t = \Phi.$$

Furthermore the following relations are valid in general, and therefore hold on Γ

(3a) $$dp = r\,dx + s\,dy,$$

(3b) $$dq = s\,dx + t\,dy.$$

Now, since p and q are known on Γ, equations (3) and (3a,b) constitute three linear equations for the determination of r,s,t on the curve. The determinant of this system is

$$\Delta = \begin{vmatrix} A & 2B & C \\ dx & dy & 0 \\ 0 & dx & dy \end{vmatrix} = A\,dy^2 - 2\,B\,dx\,dy + C\,dx^2.$$

Only when this determinant Δ is different from zero can r,s,t be calculated from (3), (3a), and (3b). However, in general, two directions, $dy:dx$, exist for every point (x,y), for which this is not the case. Therefore two (real or conjugate complex) families of curves exist on which $\Delta = 0$, and which, according to Monge, are called *characteristics*.[1] They are the dotted lines of Fig. 9. Along each of these characteristics it is in general impossible to solve for r,s,t in terms of u,p,q. We shall therefore demand as a necessary condition for the solvability of our problem, that Γ shall be *nowhere tangent to a characteristic.* The opposite case, in which Γ coincides with one of the characteristics, will be discussed in §9A in connection with D'Alembert's solution.

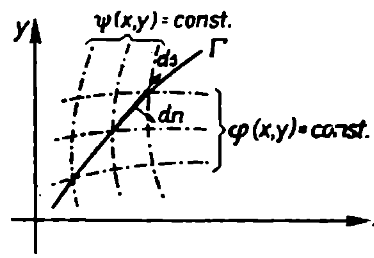


Fig. 9. The curve Γ along which u and $\partial u/\partial n$ are given, and the two families of characteristics $\xi = \varphi(x,y) = \text{const.}$ and $\eta = \psi(x,y) = \text{const.}$

When the condition $\Delta \neq 0$ is satisfied, a solution of the differential equation in the neighborhood of Γ must exist. Then the higher derivatives can be calculated in exactly the same way as the second derivatives. Let us consider, say, the third derivatives:

$$r_x = \frac{\partial^3 u}{\partial x^3}, \quad s_x = \frac{\partial^3 u}{\partial x^2 \partial y} = r_y, \quad t_x = \frac{\partial^3 u}{\partial x \partial y^2} = s_y, \quad t_y = \frac{\partial^3 u}{\partial y^3}.$$

Differentiating (3) and (3a,b) with respect to x, we get:

$$\begin{aligned} A r_x + 2\,B\,s_x + C\,t_x &= \Phi_x + \cdots \\ r_x\,dx + s_x\,dy &= dr, \\ s_x\,dx + t_x\,dy &= ds. \end{aligned}$$

[1] A geometrically intuitive introduction of characteristics is given, e.g., by B. Baule in v. VI (Partielle Differentialgleichungen) of his Mathematik des Naturforschers und Ingenieurs, Hirzel, Leipzig 1944.

On the right . . . represents terms that contain no third derivatives, and therefore contain only known quantities. The determinant of this system is again Δ. The same holds for equations obtained by differentiation with respect to y. Our condition is therefore sufficient for the computability of the third and all higher derivatives. Therefore u can be expanded in a Taylor series at every point of Γ and the coefficients are uniquely determined by the boundary conditions on Γ.

We now turn to the discussion of the equation of characteristics

(4) $$A\, dy^2 - 2\, B\, dx\, dy + C\, dx^2 = 0,$$

where we restrict ourselves to an arbitrarily chosen neighborhood in the xy-plane,[2] and distinguish between the following cases:

1) $A\,C - B^2 > 0$ *elliptic type* in which the characteristics are conjugate complex.

2) $A\,C - B^2 < 0$ *hyperbolic type* in which the characteristics form two distinct families.

3) $A\,C - B^2 = 0$ *parabolic type* in which only one real family of characteristics exists.

Each of the three types can be brought into a special normal form in which the equations of the characteristics are utilized for the introduction of new coordinates. Let these equations be

(4a) $$\varphi\,(x, y) = \text{const. and } \psi\,(x, y) = \text{const.}$$

respectively. Then through the transformation

(5) $$\xi + i\,\eta = \varphi\,(x, y), \qquad \xi - i\,\eta = \psi\,(x, y)$$

one obtains the normal form for the *elliptic* type,

(5a) $$\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} = X\left(u, \frac{\partial u}{\partial \xi}, \frac{\partial u}{\partial \eta}, \xi, \eta\right);$$

through the transformation

(6) $$\xi = \varphi\,(x, y), \qquad \eta = \psi\,(x, y)$$

one obtains the normal form for the *hyperbolic* type,

(6a) $$\frac{\partial^2 u}{\partial \xi\, \partial \eta} = X\left(u, \frac{\partial u}{\partial \xi}, \frac{\partial u}{\partial \eta}, \xi, \eta\right);$$

[2] When A, B, C depend on x, y, then the equation may obviously be of different types for different neighborhoods of the xy-plane.

and through

$$\xi = \varphi(x, y) = \psi(x, y), \quad \eta = x$$

one obtains the normal form for the *parabolic* type,

$$\frac{\partial^2 u}{\partial \eta^2} = X\left(u, \frac{\partial u}{\partial \xi}, \frac{\partial u}{\partial \eta}, \xi, \eta\right). \tag{7a}$$

Before proving this, we compare the above forms (5a), (6a) and (7a) with the equations (7.3), (7.6a), and (7.16), i.e., with the two-dimensional potential equation, the equation of the vibrating string, and the equation of linear heat conduction. We observe that the left hand sides of (5a) and (7.3) coincide except for the letters used to denote the independent variables. The analogous relation holds between (7a) and (7.16). In (6a) we only have to perform the simple transformation

$$\xi = \frac{1}{2}(\xi' + \eta'), \qquad \eta = \frac{1}{2}(\xi' - \eta') \tag{8}$$

with the inverse

$$\xi' = \xi + \eta, \quad \eta' = \xi - \eta \tag{8a}$$

we obtain

$$\frac{\partial^2 u}{\partial \xi\, \partial \eta} = \frac{\partial^2 u}{\partial \xi'^2} - \frac{\partial^2 u}{\partial \eta'^2},$$

which establishes the essential equality of the left hand sides of (6a) and (7.6a). Hence *the two-dimensional potential equation, the equation of the vibrating string and the equation of linear heat conduction are the simplest examples of the elliptic, the hyperbolic, and of the parabolic types,* respectively.

Starting with the treatment of the *hyperbolic case*, we first show that (6a) is obtained from the initial equation (2) through the transformation (6). From (6) we obtain for the first derivatives

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}\varphi_x + \frac{\partial u}{\partial \eta}\psi_x, \qquad \frac{\partial u}{\partial y} = \frac{\partial u}{\partial \xi}\varphi_y + \frac{\partial u}{\partial \eta}\psi_y$$

where the subscripts again denote differentiation. From this we obtain for the second derivatives

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2} &= \frac{\partial^2 u}{\partial \xi^2}\varphi_x^2 + 2\frac{\partial^2 u}{\partial \xi\,\partial \eta}\varphi_x\psi_x + \frac{\partial^2 u}{\partial \eta^2}\psi_x^2 + \cdots \\
\frac{\partial^2 u}{\partial x\,\partial y} &= \frac{\partial^2 u}{\partial \xi^2}\varphi_x\varphi_y + \frac{\partial^2 u}{\partial \xi\,\partial \eta}(\varphi_x\psi_y + \varphi_y\psi_x) + \frac{\partial^2 u}{\partial \eta^2}\psi_x\psi_y + \cdots \\
\frac{\partial^2 u}{\partial y^2} &= \frac{\partial^2 u}{\partial \xi^2}\varphi_y^2 + 2\frac{\partial^2 u}{\partial \xi\,\partial \eta}\varphi_y\psi_y + \frac{\partial^2 u}{\partial \eta^2}\psi_y^2 + \cdots
\end{aligned}$$

where the three dots stand for terms containing only first derivatives. Multiplying the last three equations by A, $2B$ and C, respectively, and adding, we obtain for the left side of (2):

$$
\begin{aligned}
&\frac{\partial^2 u}{\partial \xi^2}\left(A\varphi_x^2 + 2\,B\varphi_x\varphi_y + C\varphi_y^2\right)\\
(9)\qquad &+ 2\,\frac{\partial^2 u}{\partial \xi\,\partial \eta}\left(A\varphi_x\psi_x + B\left(\varphi_x\psi_y + \varphi_y\psi_x\right) + C\varphi_y\psi_y\right)\\
&+ \frac{\partial^2 u}{\partial \eta^2}\left(A\psi_x^2 + 2\,B\psi_x\psi_y + C\psi_y^2\right) + \cdots
\end{aligned}
$$

But here the coefficients of $\partial^2 u/\partial\xi^2$ and $\partial^2 u/\partial\eta^2$ vanish, since for the family of characteristics $\varphi =$ const. we have

$$\varphi_x\,dx + \varphi_y\,dy = 0,$$

Hence on introducing the ratio $dx{:}dy$ into (4) we get

$$(10)\qquad A\varphi_x^2 + 2\,B\varphi_x\varphi_y + C\varphi_y^2 = 0.$$

The derivatives of ψ must satisfy the same equation. Hence (9) indeed reduces to the hyperbolic normal form (6a) if we transfer the coefficient of $\partial^2 u/\partial\xi\partial\eta$ in (9) to the other side of the equation.

Since in the *parabolic case* we have $\eta = x$, we must substitute in (9)

$$(11)\qquad \psi(x, y) = x; \quad \text{and hence} \quad \psi_x = 1, \quad \psi_y = 0,$$

whereas (10) still holds for φ_x, φ_y. The first term in (9) therefore vanishes. Owing to (11) the coefficient of the second term reduces to $A\varphi_x + B\varphi_y$ which also vanishes since $A\,C - B^2 = 0$ makes the left side of (10) a perfect square, so that (10) can be rewritten as $(A\varphi_x + B\varphi_y)^2/A = 0$. Considering (9) and (11) the third term finally becomes simply

$$A\,\frac{\partial^2 u}{\partial \eta^2},$$

which is the parabolic normal form (7a).

The *elliptic case* need not be treated separately. It can be reduced to the hyperbolic case by a transformation analogous to (8a):

$$\xi' = \xi + i\eta, \qquad \eta' = \xi - i\eta.$$

§ 9. Differences Among Hyperbolic, Elliptic, and Parabolic Differential Equations. The Analytic Character of Their Solutions

The problem of integration, which is illustrated in Fig. 9, is applied in physics only to the case of hyperbolic differential equations; for elliptic differential equations it is replaced by an entirely different kind of problem, the *boundary value problem.* For the time being, we shall discuss this profound difference only sketchily and refer the reader to the following sections for a more precise treatment.

A. Hyperbolic Differential Equations

As the simplest example we use the equation of the vibrating string, which, written in its normal form, is

$$\frac{\partial^2 u}{\partial \xi \, \partial \eta} = 0, \quad \xi = x + y, \quad \eta = x - y, \quad y = ct. \tag{1}$$

Here the characteristics are the lines ξ = const., η = const., which in Fig. 10 are drawn at 45° angles with the x- and y-axes. The general solution of (1) is the sum of a function of ξ and a function of η:

$$u = F_1(\xi) + F_2(\eta). \tag{2}$$

Because of the meaning of ξ and η this is *d'Alembert's solution* (see V. II, §13). For the sake of simplicity, let us consider u as being given on segments AB and AD of two of the characteristics. This determines u in the entire rectangle $ABCD$. We could calculate the value of u at P by passing in the directions of the characteristics to P_1 and P_2, and substituting into (2) the values $F_1(\xi)$, $F_2(\eta)$ which are given at these points. *The values along two intersecting characteristics determine the function everywhere. For example, any discontinuities of the given functions on the characteristics would be continued into the interior of ABCD.* Thus the solution need not be an *analytic function*[3] of x and y over its domain of definition.

In physics one is given the values of u *and* of $\partial u/\partial y$ along a segment of length l on the x-axis (l = length of string):

$$u = u(x, 0) \qquad \text{and} \qquad \frac{\partial u}{\partial y} = \frac{1}{c}\frac{\partial u}{\partial t} = v(x, 0).$$

This segment corresponds to the curve Γ of Fig. 9, on which, too, u and $\partial u/\partial n$ were given, and it satisfies the requirement of not being tangent to any characteristic.

In order to apply the conclusions drawn from (2) to our present problem, we have to calculate F_1 and F_2 from our given $u(x,0)$, $v(x,0)$. This is done with the help of the following equations, which are immediate consequences of (2):

[3] A function of two real variables x, y is called analytic in a certain domain, if in some neighborhood of each point (x_0, y_0) of this domain it can be represented as a power series in $x - x_0$ and $y - y_0$.

$$u(x,0) = F_1(x) + F_2(x), \qquad F_1(x) = \frac{1}{2}\left\{u(x,0) + \int v(x,0)\,dx\right\},$$
$$v(x,0) = F_1'(x) - F_2'(x). \qquad F_2(x) = \frac{1}{2}\left\{u(x,0) - \int v(x,0)\,dx\right\}.$$

We conclude: *the given initial values, together with any possible discontinuities, are continued along the characteristics.* The solution, $u(x,y)$ is in general *not an analytic function* of x and y. It is determined *only within the rectangle of characteristics determined by the length of string* l as shown in Fig. 10.

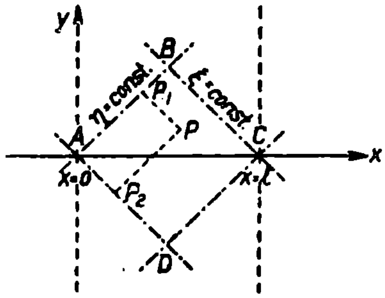


Fig. 10. The vibrating string of length l and the square of characteristics determined by its end point.

However, from a physical point of view, the solution must be determined from the initial time on, i.e., for all $y > 0$. This indicates that, in addition to the initial values, certain boundary values must be prescribed at the ends of the string. These are the stringing conditions $u = 0$ for $x = 0$ and $x = l$. Just as for all x such that $0 < x < l$ *two* values (u and $\partial u/\partial y$) had to be given, so for all $y > 0$, *two* values are given. This is due to the fact that our differential equation is of second order in both variables x,y and the only difference is that both values along the x-axis are given at the *same* point $(x,0)$ whereas the values along the y-direction are given at the different points $(0,y)$ and (l,y). The only exceptions to this rule of two necessary boundary conditions are the characteristics on which, as we saw above, *one value* (F_1 *or* F_2) *is sufficient.*

We shall show in §11 that these results, which we have established for the case of the vibrating string, can be extended to all cases of hyperbolic type.

B. Elliptic Differential Equations

Here the characteristics are imaginary and therefore have no direct bearing on the problems we are going to treat. These problems do not deal with an arc Γ, as in Fig. 9, but rather with a *closed region* S of the real xy-plane. On the boundary of S, u *or* $\partial u/\partial n$ (or a linear combination of u and $\partial u/\partial n$) will be given but not *both* u and $\partial u/\partial n$ as in the hyperbolic case. Discontinuities of the boundary values are *not* continued into the interior of S, but only into the imaginary domain, and the function u is *analytic* everywhere in the interior of S.

These are known theorems from the theory of functions (two-dimensional potential theory). Their proof for arbitrary linear elliptic differential equations will be given in the following section.

The analogue to d'Alembert's solution (2) is given in potential theory by

$$u = f_1(x + i\,y) + f_2(x - i\,y),$$

where, in order that u be real, we must set $f_2 = f_1^*$, i.e., f_2 conjugate[4] to f_1 We may also write:

$$u = \mathrm{Re}[f(z)], \tag{3}$$

where f is an arbitrary analytic function of the complex variable $z = x + iy$. However, this general solution of the equation $\Delta u = 0$ does not help us (at least not directly) in the general solution of our boundary value problem.

C. Parabolic Differential Equations

Here the two families of characteristics have degenerated into *one.* In the special case of the normal form of the equation of linear heat conduction this is the family of lines parallel to the x-axis. Only *one* boundary condition should be given on these characteristics just as in the case of hyperbolic differential equations (see p. 42). We can also see this directly from (7.16): here $\partial u/\partial y$ is determined uniquely if u is given as a function of x for some fixed y. From physical considerations one sees this in the following manner: the thermal behavior of a rod of length l is determined once and for all as soon as its initial temperature is given together with conditions for the ends of the rod (the lateral surface of the rod must be considered adiabatically closed, if heat is to flow only in the x-direction).

We shall see in §12, that the temperature distribution of the rod becomes *an analytic function of x and y* for arbitrary — even discontinuous — initial temperature. To this extent, therefore, the parabolic type resembles the elliptic type. However, the problem is not relative to a bounded region, but rather, as in the hyperbolic case, relative to a strip, i.e., a region which is infinite in one direction. The parabolic type, therefore, occupies a middle position between the elliptic and the hyperbolic types.

[4] We use the notation f^* instead of $\overline{f}$, which is more common in mathematical literature, since we want to reserve the use of the bar for mean values in time. Re and Im stand for the real and imaginary part respectively.

§10. Green's Theorem and Green's Function for Linear, and, in Particular, for Elliptic Differential Equations

In (8.1) we had the general form of a linear differential equation of second order. In order to retain a common expression for the three types, we shall not transform this system into its canonical form for the time being.

A. Definition of the Adjoint Differential Expression

We now have to introduce the seemingly rather formal concept of *the differential form $M(v)$ which is adjoint to $L(u)$*. It is defined by the requirement that the expression $v\,L(u) - u\,M(v)$ be *generally integrable* or as we may put it, that it be a *kind of divergence*.
We demand, namely

(1) $$v\,L(u) - u\,M(v) = \frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y}.$$

The problem is to determine M and X,Y as functions of v and of u,v respectively.[5]

We shall use the following identities:

(2) $$vA\frac{\partial^2 u}{\partial x^2} - u\frac{\partial^2 Av}{\partial x^2} = \frac{\partial}{\partial x}\left(Av\frac{\partial u}{\partial x} - u\frac{\partial Av}{\partial x}\right), \ldots$$

(2a) $$v\,B\frac{\partial^2 u}{\partial x\,\partial y} - u\frac{\partial^2 Bv}{\partial x\,\partial y} = \frac{\partial}{\partial x}\left(v\,B\frac{\partial u}{\partial y}\right) - \frac{\partial}{\partial y}\left(u\frac{\partial Bv}{\partial x}\right) = \cdots$$

(3) $$Dv\frac{\partial u}{\partial x} - u\frac{\partial}{\partial x}(-Dv) = \frac{\partial}{\partial x}(Duv), \ldots$$

Here the three dots (. . .) indicate the fact that (2) and (3) remain valid if we replace x by y and A,D by C,E respectively, and that on the right side of (2a) we may use the symmetric expression obtained by interchanging x and y. From this we get:

(4) $$M(v) = \frac{\partial^2 Av}{\partial x^2} + 2\frac{\partial^2 Bv}{\partial x\,\partial y} + \frac{\partial^2 Cv}{\partial y^2} - \frac{\partial Dv}{\partial x} - \frac{\partial Ev}{\partial y} + Fv,$$

(5) $$\begin{aligned} X &= A\left(v\frac{\partial u}{\partial x} - u\frac{\partial v}{\partial x}\right) + B\left(v\frac{\partial u}{\partial y} - u\frac{\partial v}{\partial y}\right) + \left(D - \frac{\partial A}{\partial x} - \frac{\partial B}{\partial y}\right)uv,\\ Y &= B\left(v\frac{\partial u}{\partial x} - u\frac{\partial v}{\partial x}\right) + C\left(v\frac{\partial u}{\partial y} - u\frac{\partial v}{\partial y}\right) + \left(E - \frac{\partial B}{\partial x} - \frac{\partial C}{\partial y}\right)uv. \end{aligned}$$

[5] The operation of divergence is properly defined only for a *vector*. Since, as equation (5) will show, X and Y are not vector components, we speak of "a kind of divergence."

Obviously X and Y are determined only up to quantities X_0, Y_0, whose divergence vanishes. We can therefore change the terms in (5): we may add $-\partial\Phi/\partial y$ to X and $+\partial\Phi/\partial x$ to Y, where Φ is an arbitrary function of x,y as well as of u,v.

We see that the relation between L and M is *reciprocal:* $L(v)$ is the adjoint differential form to $M(u)$.

Of particular importance for mathematical physics are those differential expressions for which $L(u) = M(u)$. They are called *self-adjoint.* By comparing (4) with (8.1) we get the condition of self-adjointness

$$\frac{\partial A}{\partial x} + \frac{\partial B}{\partial y} = D, \qquad \frac{\partial B}{\partial x} + \frac{\partial C}{\partial y} = E. \tag{6}$$

B. Green's Theorem for an Elliptic Differential Equation in its Normal Form

We now consider a region S with boundary curve C in the xy-plane and integrate (1) over S. We denote the element of area of S by $d\sigma$, and the line element of C by ds; let the orientation be counter-clockwise (see Fig. 11).

Applying Gauss' theorem[6] we get

$$\begin{aligned}\int_S [v\,L(u) - u\,M(v)]\,d\sigma &= \int_S \left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y}\right) d\sigma \\ &= \int_C \left\{X \cos(n, x) + Y \cos(n, y)\right\} ds.\end{aligned} \tag{7}$$

This is the *general formulation of Green's theorem* which is valid for all three types. Setting $A = C = 1$, $B = 0$, we specialize it to the case of the *elliptic type* in *normal form.* We then have:

$$\begin{aligned}\int_S [v\,L(u) - u\,M(v)]\,d\sigma &= \int_C \left(v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n}\right) ds \\ &+ \int_C \left\{D \cos(n, x) + E \cos(n, y)\right\} u\,v\,ds.\end{aligned} \tag{7 a}$$

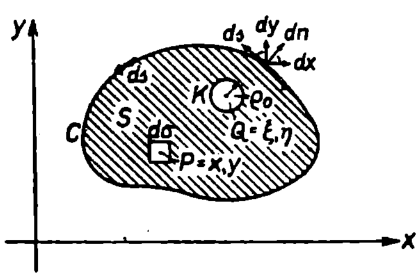


Fig. 11. Illustrating Green's theorem for an elliptic differential equation. The integration with respect to $d\sigma$ is extended over the domain S between the boundary curve C and the circle K of radius ϱ_0 which contains the unit source at Q.

[6] It states, when applied to a two-dimensional vector $\mathbf{A}$ with components X, Y, that $\int \operatorname{div} \mathbf{A}\, d\sigma = \int \mathbf{A}_n\, ds$,

This is a generalization of *Green's theorem* of potential theory

$$\int (v\,\Delta u - u\,\Delta v)\,d\sigma = \int \left(v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n}\right) ds,$$

which is obtained from (7a) by setting $D = E = 0$. (The fact that in potential theory also $F = 0$ is of no importance here.)

We shall meet another form of Green's theorem in exercise 112.

If, in the interior of S, u and v satisfy the equations

$$L(u) = 0, \quad M(v) = 0$$

then the left hand sides of the equations (7), (7a) vanish. These equations, therefore, become

$$\text{(7 b)} \qquad 0 = \int_C \{X \cos(n, x) + Y\cos(n, y)\}\, ds$$

$$\text{(7 c)} \qquad 0 = \int_C \left(v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n}\right) ds + \int_C \{D\cos(n, x) + E\cos(n, y)\}\, u\, v\, ds.$$

However, this holds only if u and v and the derivatives which appear here are *continuous* throughout S. If v has a discontinuity at the point $Q = (\xi, \eta)$, then it must be excluded from the domain of integration, just as in all applications of Green's theorem. We therefore surround Q by a curve K, which we choose to be a circle of arbitrarily small radius ρ_0. The integration in (7b,c) must then be taken over both boundaries K and C:

$$\int_K \ldots ds + \int_C \ldots ds = 0,$$

where the orientation is opposite on the two curves and the direction n is to the exterior of S.

If for K we use (7c) and for C we use (7b) we get:

$$\text{(8)} \qquad \begin{aligned} &\int_K \left(v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n}\right) ds + \int_K u\, v\,\{D\cos(n, x) + E\cos(n, y)\}\, ds \\ &\qquad = -\int_C \{X\cos(n, x) + Y\cos(n, y)\}\, ds. \end{aligned}$$

or, written in terms of coordinates:

$$\int\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y}\right) d\sigma = \int [X\cos(n, x) + Y\cos(n, y)]\, ds.$$

We apply this formally in (7) to our "pseudovector" X, Y.

C. Definition of a Unit Source and of the Principal Solution

We shall assume that the discontinuity of v at Q consists of a "*unit source.*" By this we mean the following: the yield q of a source Q is defined as the outward gradient of its field v. If we denote the distance from Q by ϱ, we have

$$q = \int_K \frac{\partial v}{\partial \varrho} ds \tag{9}$$

where K has the same meaning as before. Assuming that in the immediate neighborhood of the source v depends only on ϱ, we get

$$q = \int_{\varphi=-\pi}^{+\pi} \frac{dv}{d\varrho} \varrho \, d\varphi = 2\pi \varrho \frac{dv}{d\varrho}. \tag{9a}$$

A unit source is therefore given by:

$$\frac{dv}{d\varrho} = \frac{1}{2\pi\varrho}, \qquad v = \frac{1}{2\pi} \log \varrho + \text{const} \qquad \text{for} \quad \varrho \to 0. \tag{9b}$$

For arbitrary ϱ we write:

$$v = U \log \varrho + V, \qquad \varrho = \sqrt{(x-\xi)^2 + (y-\eta)^2}, \tag{10}$$

where U and V are analytic functions of x, y and ξ, η such that U becomes $1/2\pi$ for $(x, y) \to (\xi, \eta)$.

A function of this kind we call a *principal solution* of the differential equation $M(v) = 0$. In the same way we shall speak of a principal solution of the adjoint equation $L(u) = 0$. Since the latter also corresponds to a unit source it will have the same form (10), although in general U and V will be different functions. Here too we can assume U and V to be analytic as long as the coefficients D, E, F in the differential equation are analytic. In the case of the potential equation $\Delta u = 0$ our principal solution corresponds essentially to the logarithmic potential, where we have for all ϱ

$$v = \frac{1}{2\pi} \log \varrho. \tag{10a}$$

D. The Analytic Character of the Solution of an Elliptic Differential Equation

We return now to equation (8). Substituting (10) in (8), we see that only the term with

$$\frac{\partial v}{\partial n} = -\frac{\partial v}{\partial \varrho} = -\frac{U}{\varrho} + \cdots = -\frac{1}{2\pi\varrho} + \cdots$$

contributes to the integral over K, while all the other terms on the left side of (8) have zeros of the same order as $\varrho \log \varrho$ or of higher order. Since u is continuous at Q and the perimeter of K is $2\pi\varrho_0$, we obtain for the left side of (8):

$$-\int_K u \frac{\partial v}{\partial n} ds = \frac{u_Q}{2\pi\varrho_0} \int_K ds = u_Q;$$

and equation (8) becomes

$$u_Q = -\int_C \{X \cos(n, x) + Y \cos(n, y)\} ds. \tag{11}$$

The most interesting aspect of this formula is its dependence upon ξ, η which is brought about by the terms v, $\partial u/\partial x$, $\partial v/\partial y$ that enter in X and Y and are given analytically by (10). When Q lies in the *interior* of S (*not* on the boundary), then $\log \varrho$ is a regular analytic function, since the point $P = (x,y)$ in the integration is restricted to the boundary curve C and does not coincide with Q. *Therefore* $u_Q = u(\xi, \eta)$ *is an analytic function of* ξ,η *in the interior of* S. This holds whether or not the boundary values u, $\partial u/\partial x$, $\partial u/\partial y$ are analytic; in any case the dependence of the integrand on x,y disappears upon integration with respect to ds. Even discontinuities of the boundary values are averaged out. *Discontinuities on the boundary are not continued into the interior of S.* (The characteristics are imaginary.) This proves the assertion of §9B.

For a self-adjoint differential equation in its normal form we have, according to (6), $D = E = 0$. Using the form (7c) of the line integral we get from (11)

$$u_Q = \int_C \left(u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n}\right) ds. \tag{11a}$$

Using expression (10a) for v, we get for the special case of the *potential equation:*

$$u_Q = u(\xi, \eta) = \frac{1}{2\pi} \int_C \left(u \frac{\partial \log \varrho}{\partial n} - \log \varrho \frac{\partial u}{\partial n}\right) ds. \tag{11b}$$

E. The Principal Solution for an Arbitrary Number of Dimensions

We restrict ourselves here to the case of the potential equation. The three-dimensional analogue of (9b) is

$$\frac{\partial v}{\partial r} = \frac{1}{4\pi r^2}, \qquad v = -\frac{1}{4\pi r} + \text{const.}$$

(r = distance from the source at Q, $4\pi r^2$ = surface area of the sphere.) This is essentially[7] the so-called "Newtonian potential."

In the four-dimensional case we have equation (7.5). This yields the principal solution:

$$\frac{\partial v}{\partial R} = \frac{1}{2\pi^2 R^3}, \qquad v = -\frac{1}{4\pi^2 R^2} + \text{const.}$$

where R is the distance from Q and $2\pi^2 R^3$ is the surface of the hypersphere. The following table shows the decreasing orders of infinity at the source with decreasing number of dimensions. For the dimension one v is continuous at the source. In fact, the potential equation in one dimension is $d^2v/dx^2=0$ which yields $dv/dx=$ const. The constant will have different values C_1 and C_2 on the right and left side of the source respectively. This follows from the condition that it be a unit source so that $C_1 - C_2 = 1$. The discontinuity has passed from v to the gradient of v. (See exercise II.3).

Dimension	4	3	2	1
grad v . . .	$\frac{1}{2\pi^2 R^3}$	$\frac{1}{4\pi r^2}$	$\frac{1}{2\pi\varrho}$	C_1 or C_2
v	$-\frac{1}{4\pi^2 R^2}$	$-\frac{1}{4\pi r}$	$-\frac{1}{2\pi}\log\frac{1}{\varrho}$	continuous

F. Definition of Green's Function for Self-Adjoint Differential Equations

We now deal with the boundary value problem of §9. This question is by no means settled by the construction of the principal solution. We first consider the simplest case of self-adjointness. In order to calculate u at the point Q in equation (11a) we must know *both u and $\partial u/\partial n$* on C, whereas in the boundary value problem we are given *either u or $\partial u/\partial n$*.

[7] The denominator 4π corresponds to the "rational units" of electrodynamics.

Our problem is now to modify the principal solution v, so as to eliminate $\partial u/\partial n$ (or u) from (11a). We call this modified function of the two pairs of variables x,y and ξ, η Green's function and denote it by $G(P,Q)$. It has to satisfy the following conditions:

a) $L(G) = 0$ in the interior of C,

b) $G = 0$ (or $\partial G/\partial n = 0$) on C,

c) $\lim\limits_{P \to Q} G(P, Q) \to \frac{1}{2\pi} \log \varrho$ (condition of unit source).

Conditions a) and c) are the same as for the original v, but condition b) has been added. Replacing v by G in (11a) we get

$$u_Q = \int u \frac{\partial G}{\partial n} ds \qquad \text{or} \qquad \left(u_Q = -\int \frac{\partial u}{\partial n} G \, ds \right). \tag{12}$$

This solves the boundary value problem in both cases (for given u or $\partial u/\partial n$). However, due to condition b), the construction of G itself requires the solution of a boundary value problem. But this problem is simpler than the general boundary value problem, and we shall see that in special cases it can be solved in an elegant way with the help of a reflection process. On the other hand G, unlike u, is not regular in the interior of C, but like v is a function with a prescribed unit source.

Equation (12) reduces the boundary value problem to a simple quadrature. Green's function plays the same role in the general theory of integral equations. It is called there the "resolving kernel."

Another interesting property of G which follows from the conditions a),b),c) is the *reciprocity relation*

d) $G(P,Q) = G(Q, P)$.

It expresses the interchangeability of *source-point and action-point,* so to speak, the interchangeability of *cause and effect.*

In order to prove d) we substitute in (7a)

$$M = L, \quad u = G(I, P), \quad v = G(I, Q).$$

The point $I = (x_1,y_1)$ shall be called "point of integration." Since u becomes infinite for $I = P$, and v for $I = Q$, these points must be excluded from the integration by infinitesimal circles K_P and K_Q. According to a), integration over the region bounded by these circles and by C makes the left side of (7a) equal to 0; also, according to b), the integral over C on the right side of (7a) becomes equal to 0. There only remain the line integrals over K_P and K_Q which, according to c), yield:

$$u_Q \int_{K_Q} \frac{ds}{2\pi \varrho_Q} - v_P \int_{K_P} \frac{ds}{2\pi \varrho_P} = G(Q, P) - G(P, Q).$$

Since this must vanish d) is proved.

Equation (12) is the solution of the boundary problem for the homogeneous equation $L(u) = 0$. We now consider the solution of the non-homogeneous differential equation

(13) $$L(u) = \varrho$$

where $\varrho(x, y)$ is an arbitrary continuous point-function in S with continuous first and second derivatives. Substituting $v = G(P, Q)$ in (7a) we get for the first term on the left side:

$$\int_S \varrho\, G(P, Q)\, d\sigma_P$$

which is added to the term in (12). Instead of (12) we get

(13a) $$u_Q = \int_S \varrho\, G\, d\sigma + \int_C u \frac{\partial G}{\partial n}\, ds$$

or, if $\partial u/\partial n$ instead of u is given on C:

(13b) $$u_Q = \int_S \varrho\, G\, d\sigma - \int_C \frac{\partial u}{\partial n} G\, ds.$$

These formulas apply to every self-adjoint differential expression in its normal form $L(u) = \Delta u + Fu$, in particular to the ordinary wave equation ($F = k^2 =$ Const.) and to the potential equation ($F = 0$).

In the case of a *non-self-adjoint* differential form $L(u)$ equations (12) and (13*a*,*b*) remain valid. But, as we see from (7a), G must satisfy the adjoint equation $M(G) = 0$ in the variables x,y; also, condition b) must be changed somewhat. Instead of a) and b) we now have:

a′) $$M(G) = 0,$$

b′) $$G = 0 \ \left(\text{or } \frac{\partial G}{\partial n} - G\{D \cos(n, x) + E \cos(n, y)\} = 0\right).$$

Condition c) remains valid. However the reciprocity law d) now reads

d′) $$G(P, Q) = H(Q, P).$$

Here H is Green's function for the *adjoint* equation to $M = 0$, and hence is satisfies the equation $L(H) = 0$ in the coordinates of Q.

§11. Riemann's Integration of the Hyperbolic Differential Equation

The normal form of a linear differential equation of second order of hyperbolic type is obtained from (8.1) by setting $A = C = 0$, $B = 1/2$:

$$L(u) = \frac{\partial^2 u}{\partial x\,\partial y} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + F u = 0. \tag{1}$$

Its adjoint differential equation is according to (10.4):

$$M(v) = \frac{\partial^2 v}{\partial x\,\partial y} - \frac{\partial D\,v}{\partial x} - \frac{\partial E\,v}{\partial y} + F v = 0. \tag{2}$$

At the same time one obtains from (10.5)

$$\begin{aligned} X &= \frac{1}{2}\left(v\frac{\partial u}{\partial y} - u\frac{\partial v}{\partial y}\right) + D\,u\,v,\\ Y &= \frac{1}{2}\left(v\frac{\partial u}{\partial x} - u\frac{\partial v}{\partial x}\right) + E\,u\,v. \end{aligned} \tag{3}$$

Substituting (1),(2),(3) in (10.7) we get:

$$\int_S \{v\,L(u) - u\,M(v)\}\,d\sigma = \int_C \{X\cos(n, x) + Y\cos(n, y)\}\,ds. \tag{4}$$

In order to obtain an integration of the hydrodynamic equations Riemann chose as the region S the "triangle" PP_1P_2 of Fig. 12, with a boundary consisting of the segments of characteristics PP_1 and PP_2 and of the arc P_1P_2 of the curve Γ. u and $\partial u/\partial n$ are given on Γ, which implies that $\partial u/\partial x$, $\partial u/\partial y$ are given on Γ (see p. 36). The curve Γ must satisfy the condition that it be tangent to no characteristic (p. 37). The function v in (4) is determined according to Riemann by the conditions:

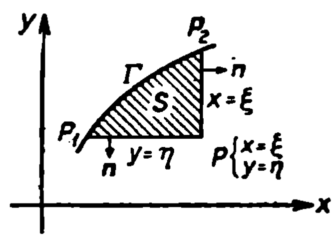


Fig. 12. Riemann's integration of a hyperbolic differential equation in its normal form with the help of the characteristic function v.

(5a) $M(v) = 0$ in S with respect to the variables x,y;

(5b) $v = 1$ at the point P with coordinates $x = \xi$, $y = \eta$;

(5c) $\frac{\partial v}{\partial y} - D\,v = 0$ on the characteristic $x = \xi$,

$\frac{\partial v}{\partial x} - E\,v = 0$ on the characteristic $y = \eta$.

We add the following remarks.

1. It would not be possible to replace (5b) by a condition of discontinuity as in (10.10), since a hyperbolic equation does not admit isolated singularities (every singularity is continued along the characteristics). For this reason we no longer call v the "principal solution" or "Green's function" but call it the "characteristic function."

2. The conditions (5c) prescribe only *one* condition each on the characteristics $x = \xi$ and $y = \eta$, whereas on Γ two conditions have to be given for u. This corresponds to the fact that the characteristics are an exception with respect to the boundary conditions that are to be prescribed on them; we saw this in §9 in connection with the equation of the vibrating string. If we call the boundary value problem along two intersecting characteristics a *boundary value problem of the second kind,* in contrast to a *boundary value problem of the first kind* along a general curve Γ, then we can say that Riemann's method consists of *the reduction of a boundary value problem of the first kind to a much simpler boundary value problem of the second kind.*

Substituting condition (5a) in (4) and remembering that $L(u) = 0$ we get

$$0 = \int_{\Gamma} \cdots + \int_{P_1}^{P} \cdots + \int_{P}^{P_2} \cdots \tag{6}$$

In the last integral $\cos(n,y) = 0$ and only the X term remains. We transform the term with $\partial u/\partial y$ by integration by parts:

$$\frac{1}{2}\int_{P}^{P_2} v\frac{\partial u}{\partial y}dy = \frac{1}{2}v\,u\Big|_{P}^{P_2} - \frac{1}{2}\int_{P}^{P_2} u\frac{\partial v}{\partial y}dy.$$

Combining this with the other terms we get

$$\int_{P}^{P_2} X\,dy = \frac{1}{2}(v\,u)_{P_2} - \frac{1}{2}(v\,u)_P - \int_{P}^{P_2} u\left(\frac{\partial v}{\partial y} - Dv\right)dy. \tag{6a}$$

For the middle integral of (6) where $\cos(n,x) = 0$ and $\cos(n,y) = -1$ (n is the outer normal) we get in an analogous manner:

$$-\int_{P_1}^{P} Y\,dx = \frac{1}{2}(v\,u)_{P_1} - \frac{1}{2}(v\,u)_P + \int_{P_1}^{P} u\left(\frac{\partial v}{\partial x} - E\,v\right)dx. \tag{6b}$$

The integrals on the right sides of (6a) and (6b) vanish on account of condition (5c). If we consider (5b) equation (6) becomes,

$$(7)\qquad u_P = \int_\Gamma \{X \cos(n, x) + Y \cos(n, y)\}\, ds + \tfrac{1}{2}\{(v\,u)_{P_1} + (v\,u)_{P_2}\}.$$

The value of u at an arbitrary point P is given here in terms of the values of u and its first derivatives on Γ as they enter in X and Y (u_{P_1} and u_{P_2} are among those values). We state: Equation (7) reduces the boundary value problem of the *first* kind for u to the problem of the computation of v, that is to a boundary value problem of the *second* kind which is given by the conditions (5a,b,c).

The computation of v is not difficult. It is particularly easy in the hydrodynamic example that was treated by Riemann. In that case we have[8]

$$(8)\qquad D = E = -\frac{a}{x+y}, \qquad F = 0.$$

Condition (5c) implies

$$x = \xi: \qquad \frac{1}{v}\frac{\partial v}{\partial y} = -\frac{a}{\xi + y}, \qquad v = C_1(\xi + y)^{-a},$$

$$y = \eta: \qquad \frac{1}{v}\frac{\partial v}{\partial x} = -\frac{a}{x + \eta}, \qquad v = C_2(x + \eta)^{-a}.$$

Both these conditions and condition (5b) are satisfied if we set:

$$(9)\qquad C_1 = C_2 = (\xi + \eta)^a, \qquad v = \left(\frac{\xi + \eta}{x + y}\right)^a.$$

In order to satisfy (5a) Riemann modifies (9) as follows:

$$(10)\qquad v = \left(\frac{\xi + \eta}{x + y}\right)^a F(a + 1, -a, 1, z), \qquad z = -\frac{(x - \xi)\,(y - \eta)}{(x + y)\,(\xi + \eta)},$$

where

$$(10a)\qquad F(\alpha, \beta, \gamma, z) = 1 + \frac{\alpha\beta}{\gamma}\frac{z}{1!} + \frac{\alpha(\alpha + 1)\beta(\beta + 1)}{\gamma(\gamma + 1)}\frac{z^2}{2!} + \cdots$$

is the hypergeometric series. In §24D we shall see some of the function-theoretic properties of this series, and in appendix II of Chapter IV we shall prove that v in (10) satisfies condition (5a), in other words, that it is a solution of the equation $M(v) = 0$. We note that on the characteristics we have $x = \xi$ or $y = \eta$, and therefore $z = 0$ and $F = 1$, which makes (10) identical with (9). Equation (10) for v and equation (7) for u solve our hyberbolic boundary value problem completely.

[8] The constant a which enters in (8) is expressed simply in terms of the exponent in the anisotropic equation of state from which the hydrodynamic problem is derived.

§12. Green's Theorem in Heat Conduction. The Principal Solution of Heat Conduction

The differential equation (7.14) of heat conduction

(1) $$L(u) = \frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial y} = 0, \quad y = kt,$$

is *not* self-adjoint. The adjoint equation of (1) is:

(2) $$M(v) = \frac{\partial^2 v}{\partial x^2} + \frac{\partial v}{\partial y} = 0.$$

We see this from (9.4) if we substitute the values of (1)

$$B = C = D = F = 0, A = 1, E = -1;$$

from (9.5) we get

(3) $$X = v\frac{\partial u}{\partial x} - u\frac{\partial v}{\partial x}, \quad Y = -uv.$$

Just as in the elliptic and the hyperbolic case we get *Green's theorem for linear heat conduction* from (1),(2),(3) by integration over the interior and the boundary of a bounded domain in the x,y-plane. However, since x represents a spatial measurement and y a time measurement, we shall not consider here a region with curved boundary, but only such regions whose boundaries consist of segments parallel to the x- or y-axis, as that shown in Fig. 13.

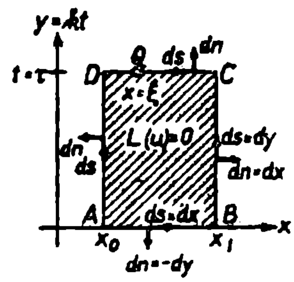


Fig. 13. Reduction of the general boundary value problem of heat conduction to the principal solution V for a rod with endpoints x_0 and x_1. The unit heat pole is at Q and has the coordinates $x = \xi$, $y = \eta$.

Along the side AB of the figure we have $ds = dx$, $dn = -dy$, $\cos(n,x) = 0$, $\cos(n,y) = -1$ and therefore

$$\int_A^B \{X\cos(n,x) + Y\cos(n,y)\}\,ds = -\int_A^B Y\,dx.$$

The same thing holds for the side CD which is also parallel to the x-axis and where the signs of both dx and $\cos(n,y)$ are reversed. Correspondingly we have for the sides BC and AD parallel to the y-axis

$$\int_B^C \{X\cos(n, x) + Y\cos(n, y)\}\, ds = +\int_B^C X\, dy\,.$$

Using the values of L,M,X,Y given in (1),(2),(3) we get the following form of Green's theorem:

$$(4)\qquad \begin{aligned} &\int\left\{v\left(\frac{\partial^2 u}{\partial x^2}-\frac{\partial u}{\partial y}\right)-u\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial v}{\partial y}\right)\right\}dx\,dy \\ &=\int u\,v\,dx+\int\left(v\frac{\partial u}{\partial x}-u\frac{\partial v}{\partial x}\right)dy\,, \end{aligned}$$

where the first integral on the right side is taken over the two sides of the rectangle that are parallel to the x-axis and the second integral is taken over the other sides.

Formula (4) also represents Green's theorem for *two-dimensional* or *three-dimensional* heat conduction if we perform the following replacements:

$$(5a)\qquad dx \quad \text{by} \quad \begin{cases} d\sigma & \text{(two-dimensional case)} \\ d\tau & \text{(three-dimensional case)} \end{cases}$$

$$(5b)\qquad \frac{\partial u}{\partial x},\ \frac{\partial v}{\partial x},\ dy \quad \text{by} \quad \frac{\partial u}{\partial n},\ \frac{\partial v}{\partial n},\begin{cases} dy\ ds & \text{(two-dimensional case)} \\ dy\ d\sigma & \text{(three-dimensional case)} \end{cases}$$

$$(5c)\qquad \frac{\partial^2 u}{\partial x^2},\ \frac{\partial^2 v}{\partial x^2} \quad \text{by} \quad \Delta u,\ \Delta v\,.$$

In the three-dimensional case we integrate over a four-dimensiona cylinder whose base is the three-dimensional heat conductor and whos generatrix is parallel to the time-axis; the integration in the second tern on the right, which is indicated in (5b) by dy and is now replaced b integration with respect to $dy\, d\sigma = k\, dt\, d\sigma$, is extended over the three dimensional lateral surface of this cylinder.

Before we apply these general formulas we must decide how w want to define the analogue of the "principal solution" of (10.9). We shal see that the "unit source" will have to be replaced by a "heat pole o strength one."

We first consider the case of linear heat conduction and its differen rial equation $L = 0$; the passage to the adjoint equation $M = 0$ and t the two-and three-dimensional cases will then be easy.

Let the heat conductor be infinite in both directions and let it temperature for $t = 0$ be given as a function of x:

$$u = f(x), \quad -\infty < x < +\infty.$$

We represent $f(x)$ by a Fourier integral as in (4.8):

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\omega \int_{-\infty}^{+\infty} f(\xi)\, e^{i\omega(x-\xi)} d\xi. \tag{6}$$

In order to obtain a solution of equation (1) we must merely complete $\exp[i\,\omega(x-\xi)]$ to the product

$$\varphi(y)\, e^{i\omega(x-\xi)}. \tag{7}$$

Substituting this in (1) we get:

$$-\omega^2 \varphi(y) = \frac{d\varphi(y)}{dy}, \qquad \varphi(y) = C\, e^{-\omega^2 y}.$$

Here $C = 1$ on account of the obvious condition $\varphi(0) = 1$. We therefore replace in (6)

(7 a) $\qquad \exp[i\,\omega(x-\xi)] \qquad$ by $\qquad \exp[i\,\omega(x-\xi) - \omega^2 y]$.

This seemingly complicates the Fourier integral (6) but in reality it makes it much simpler. In (6) $f(x)$ must converge to 0 "sufficiently rapidly" in order that the integral with respect to ξ will converge, and this integration must be performed *before* the much simpler integration with respect to ω, which would otherwise not converge. But now the order of integration is reversible and $f(x)$ is less restricted in its behavior at infinity. The new factor $\varphi(y) = \exp(-\omega^2 y)$ serves as convergence factor[9] for all $y > 0$.

Combining (6) and (7) and substituting $y = kt$ we get:

$$u(x,t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(\xi)\, d\xi \int_{-\infty}^{+\infty} e^{-\omega^2 kt + i\omega(x-\xi)} d\omega. \tag{8}$$

We abbreviate the exponent in (8) by $-\alpha\,\omega^2 + \beta\,\omega$ and complete the square:

$$-\alpha\,\omega^2 + \beta\,\omega = -\alpha\left(\omega - \frac{\beta}{2\alpha}\right)^2 + \frac{\beta^2}{4\alpha}.$$

[9] See the author's dissertation, Königsberg 1891: "Die willkürlichen Funktionen der mathematischen Physik" where the general case of the limit for $t \to 0$ of a Fourier integral with a convergence factor is considered. The function $f(x)$ may then have, for example, "an infinity of maxima and minima" or arbitrary discontinuities.

Substituting $p = \omega - \beta/2\alpha$: we get:

(9) $$\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-\omega^2 kt + i\omega(x-\xi)}\,d\omega = \frac{1}{2\pi}\, e^{-\frac{(x-\xi)^2}{4kt}}\int_{-\infty}^{+\infty} e^{-\alpha p^2}dp\,.$$

We have the well known formula for the Laplace integral:

$$\int_{-\infty}^{+\infty} e^{-p^2}dp = \sqrt{\pi}\,, \quad \text{and therefore} \quad \int_{-\infty}^{+\infty} e^{-\alpha p^2}dp = \sqrt{\frac{\pi}{\alpha}}\,.$$

Denoting the right side of (9) by U we get

(10) $$U = \frac{1}{\sqrt{4\pi kt}}\, e^{-\frac{(x-\xi)^2}{4kt}}$$

Equation (8) then becomes

(10a) $$u\,(x, t) = \int_{-\infty}^{+\infty} f(\xi)\, U\, d\xi:$$

We note that the initial temperature at the point $x = \xi$ spreads in space-time independently of the initial temperature at all other points. (This is due to the linearity of the differential equation which permits the superimposition of solutions.) For $t \to 0$ we have $u(x,t) \to f(x)$ and (10a) becomes:

$$f(x) = \int_{-\infty}^{+\infty} f(\xi)\, U\, d\xi\,.$$

This shows that U has the "character of a δ function." As on p. 27 this means that U vanishes in the limit $t \to 0$ for all values of $x \neq \xi$ and becomes infinite at $x = \xi$ so that

(10b) $$\int_{x-\epsilon}^{x+\epsilon} U\, d\xi = 1$$

(These properties of U are easily seen from (10).) Ignoring the distinction between heat-energy and temperature we may say that U describes the space-time behavior of a *unit heat-source* or of a *heat-pole of strength* 1.

For the case of a general initial time $t = \tau$ we get instead of (10):

(10c) $$U = \{4\pi k\,(t-\tau)\}^{-\frac{1}{2}}\ \exp\left\{-\frac{(x-\xi)^2}{4k\,(t-\tau)}\right\}$$

For the special case of a heat-pole at $\xi = 0$ we get

$$(10\text{d}) \qquad U = (4\pi kt)^{-\frac{1}{2}} \exp\left\{-\frac{x^2}{4kt}\right\}.$$

Before discussing the deeper meaning of these formulas we shall generalize them to two and three dimensions.

We noted the possibilities of generalizing Fourier's double integral to quadruple and sextuple integrals at the end of §4. We perform this generalization by writing instead of (6):

$$(11) \qquad f(x, y) = \frac{1}{2\pi}\int d\omega \int f(\xi, y)\, e^{i\omega(x-\xi)}\, d\xi$$

and

$$(11\text{a}) \qquad f(\xi, y) = \frac{1}{2\pi}\int d\omega' \int f(\xi, \eta)\, e^{i\omega'(y-\eta)}\, d\eta;$$

Combining (11) and (11a) we get

$$(11\text{b}) \qquad f(x, y) = \frac{1}{(2\pi)^2}\int d\omega \int d\omega' \iint f(\xi, \eta)\, e^{i\omega(x-\xi) + i\omega'(y-\eta)}\, d\xi\, d\eta.$$

The same process which led from (6) to (10a) leads for the two-dimensional case from (11b) to

$$(12) \qquad u(x, y, t) = \iint\limits_{-\infty}^{+\infty} f(\xi, \eta)\, U\, d\xi\, d\eta$$

where U is the product of two factors of the form (10):

$$(13) \qquad U = (4\pi k t)^{-1}\, e^{-\frac{(x-\xi)^2 + (y-\eta)^2}{4kt}}$$

In the three-dimensional case U is the product of three factors of the form (10):

$$(14) \qquad U = (4\pi k t)^{-\frac{3}{2}}\, e^{-\frac{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2}{4kt}}$$

Equations (13) and (14) stand for *unit heat-poles in the plane and in space*. Equations (10), (13), (14) indicate the connection between *heat-conduction* and *probability*.

We compare (10d) with the Gaussian law of error

$$(15) \qquad dW = \sqrt{\frac{\alpha}{\pi}}\, e^{-\alpha x^2}\, dx.$$

Here dW is the probability of an error between x and $x + dx$ in a measuring process whose precision is given by the "precision factor" α. In our case this factor is $(4kt)^{-1}$; "infinite precision" is given by $t = 0$ which means absolute concentration of heat in the point $x = 0$; "decreasing precision" corresponds to increasing t. Fig. 14 shows the well known bell-shaped curves which for decreasing α give the behavior of U for increasing t. The function U in (10d) is equal to the "probability density" dW/dx.

In an analogous manner we compare (13) to the measuring of a position in the xy-plane whose exact position is given by (ξ, η); and (14) with the measurement of a space point with the true position (ξ, η, ζ). The precision-factor in both cases is $\alpha = (4kt)^{-1}$ as before. This suggests that the physical reason for heat-conduction is of a *statistical* — not *dynamical* — nature. This becomes apparent in the treatment of heat conduction in the kinetic theory of gases (or, more correctly, the "statistical theory of gases"). Connected with this is the following fact which we discuss for the spatial case of equation (14). For $t = 0$ the total heat-energy is concentrated at the point (ξ, η, ζ), but after an arbitrarily short time we have a non-vanishing temperature U at a distant point (x,y,z). Hence *heat expands with infinite velocity.* This is impossible from the point of view of dynamics where no velocity may exceed c.

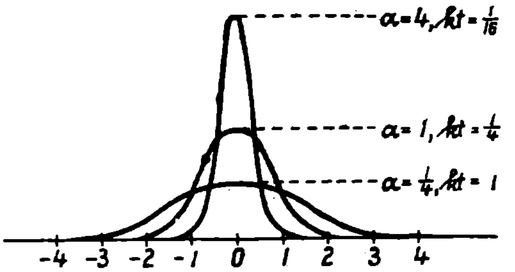


Fig. 14. The Gaussian error curve for the precision factors $\alpha = 4,1,1/4$, being at the same time the principal solution of heat conduction for $kt = 1/16,1/4,1$.

From §7, p. 34 we know that diffusion, electric conduction, and viscosity satisfy the same differential equation as heat conduction. Here too the statistical approach is clear. Diffusion is based on the *Brownian motion* in a solvent of the individual dissolved molecules, and the statistical origin is ascertained both by theory and experiment. The electron theory of metals shows that upon electrical conduction the electrons are diffused through the grid of metal molecules etc.

The function U of the equations (10), (13), (14) is the *principal solution of the differential equation* $L(u) = 0$. We now wish to transform it into the principal solution V of the *adjoint equation* $M(v) = 0$. Comparing (1) and (2) we see that this is done by reversing the sign of $y = kt$; we shall also reverse the sign of $y_0 = k\tau$ so that the heat pole will again be situated at the point $x = \xi$, $t = \tau$. Thus we obtain from (10c):

(16) $$V = \{4\pi k(\tau - t)\}^{-\frac{1}{2}} \exp\left\{-\frac{(x-\xi)^2}{4k(\tau - t)}\right\}$$

V has an essential singularity for $t = \tau$ and is defined only for the past of τ, i.e., for $t < \tau$, in contrast to the principal solution of U which is regular only for the future of τ, i.e., for $t > \tau$.

We return to Green's theorem (4). Setting $v = V$ and taking for u a solution of $L(u) = 0$ we get:

(17) $$\int u\,V\,dx + k\int\left(V\frac{\partial u}{\partial x} - u\frac{\partial V}{\partial x}\right)dt = 0\,.$$

The two integrals are extended over the sides of the rectangle of Fig. 13, the first over the two horizontal sides, the second over the two vertical sides.

Since V too is a "δ function" the first integral taken over the side $t = \tau$ yields $-u_Q$. If we decompose the second integral into the two components which correspond to the two rod ends x_0 and x_1 and denote this by $\Sigma^{(x_1,\,x_0)}$ we get:

(18) $$u_Q = \int_{x_0}^{x_1} u\,V_0\,dx + k\sum\nolimits^{(x_1,\,x_0)}\int_0^{\tau}\left(V\frac{\partial u}{\partial x} - u\frac{\partial V}{\partial x}\right)dt\,.$$

Here V_0 is V for $t = 0$.

This representation of u is general since the source point Q can be situated at the *arbitrary* point $x = \xi$, $t = \tau$. However it does not yet solve the boundary value problem of §9C, since in addition to the initial values of u it assumes that the boundary values of u *and* of $\partial u/\partial x$ are given at the endpoints, whereas in the boundary value problem only u *or* $\partial u/\partial x$ may be prescribed. In order to solve the boundary value problem we must replace V in (18) by *Green's function* G which *satisfies the condition* $G = 0$ *at the endpoints* and thereby makes the term containing $\partial u/\partial x$ in (18) vanish. We shall see in the next chapter how G can be constructed from V by a reflection process. Exercise II.4 contains an application of (18) to laminar fluid friction.

The above considerations can be transferred immediately to the two- and three-dimensional cases. As remarked above in connection with (5a, b), we merely have to extend the integration in the first integral of (18) over the base, and in the second integral over the lateral surface of the three- or four- dimensional space-time cylinder. However the construction of Green's function G by a reflection process for the two and three dimensional problems will succeed only in exceptional cases (see §17).

On the other hand equation (18) (in terms of V, not G) suffices to insure the analytic character of u, since the coordinates ξ, τ (or ξ, η, τ or ξ, η, ζ, τ) of Q appear on the right side only in the principal solution V, that is, only in analytic form. The solutions of our parabolic differential equation are *analytic* in the interior of their domain just as in the elliptic case. However, the domain here is not bounded as in the elliptic case, but is an infinite strip (as was pointed out at the end of §9). From this latter point of view the parabolic boundary value problem resembles the hyperbolic one.

Chapter III

Boundary Value Problems in Heat Conduction

§13. Heat Conductors Bounded on One Side

In the preceding section we treated the equalization process for a linear heat conductor that is infinite in both directions and represented it by equation (12.10a):

$$(1) \qquad u(x,t)=\int_{-\infty}^{+\infty} f(\xi)\,U\,d\xi, \qquad U=(4\pi kt)^{-\frac{1}{2}}\exp\left\{-\frac{(x-\xi)^2}{4kt}\right\}.$$

By the substitution

$$(1a) \qquad \xi = x+\sqrt{4kt}\,z$$

it goes over into the *Laplace form*

$$(2) \qquad u(x,t)=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty} f(x+\sqrt{4kt}\,z)\,e^{-z^2}dz\,.$$

It is instructive to compare this with d'Alembert's solution (9.2): in the latter we have two arbitrary functions F_1, F_2, corresponding to the hyperbolic type of the wave equation, whereas in (1) and (2) we have *one* arbitrary function f corresponding to the parabolic type of the equation of heat conduction.

In the case of a heat conductor which is bounded on one side, $0<x<\infty$, we have to deal first with the boundary condition at $x = 0$:

a) A given temperature $u(0,t)$; in particular the *isothermal* boundary condition

$$(3a) \qquad u = 0.$$

b) A given heat flow $\mathbf{G}\,(0,t)$ (notation as in §7, equations (9) to (12)); in particular the *adiabatic* boundary condition

$$(3b) \qquad \frac{\partial u}{\partial x} = 0.$$

c) A linear combination of both which takes into consideration the so-called *outer heat conduction*, written in the conventional form

(3c) $$\frac{\partial u}{\partial n} + hu = 0.$$

Here n stands for the *outer* normal which in our case is in the direction of the *negative* x-axis. The name "outer heat conduction" summarizes the effect of convection, the radiation into the surrounding medium and the heat conduction into that medium which is usually negligible. We note that (3c) is obtained as an approximation of the radiation law of Stefan-Boltzmann which states: the radiation of heat per unit of time and area of a body of absolute temperature T is proportional to T^4. If we denote the factor of proportionality by a and if the end of the rod[1] is in a neighborhood of temperature T_0 which radiates towards the end of the rod an amount of heat aT_0^4 per unit of time and area, then the energy emitted in the normal direction by a surface element $d\sigma$ of the end of the rod is given by:

$$dQ_n = a(T^4 - T_0^4)\,d\sigma\,dt.$$

Since usually both temperatures T and T_0 are far from absolute zero, we get

(4) $$dQ_n \sim 4\,a\,T_0^3\,u\,d\sigma\,dt \quad \text{with} \quad u = T - T_0 \ll T \quad \text{and} \quad \ll T_0.$$

This amount of heat dQ_n must be balanced by the heat flow from the interior of the rod which is given by Fourier's law (7.12). Hence we write:

(4a) $$dQ_n = -\varkappa \frac{\partial u}{\partial n}\,d\sigma\,dt.$$

By comparison of (4) and (4a) we obtain:

$$4\,a\,T_0^3\,u = -\varkappa\frac{\partial u}{\partial n},$$

and hence

(5) $$\frac{\partial u}{\partial n} + h\,u = 0, \qquad h = \frac{4\,a\,T_0^3}{\varkappa},$$

which corresponds to (3c) and shows h to be a *positive* constant.

We first treat conditions (3a) and (3b). These conditions are satisfied if we develop f, which is given only for $0 < x < \infty$, into a *pure sine or*

[1] We speak here of a "rod" although the linear heat conductor need not have the form of a thin rod but may have an arbitrary cross-section so long as its state depends only on *one* coordinate; for a real rod one must add the adiabatic condition $\partial u/\partial n = 0$ for the lateral surface (see the end of §16).

cosine integral, or in other words, if we continue f in the negative x-axis as an *even* or *odd* function (see equations (4.11a,b)). If as in (12.8) we append the time dependence factor $\exp(-\omega^2 k t)$ which is required by the equation of heat conduction, and integrate with respect to ω, then (1) becomes

$$u(x, t) = \int_0^\infty f(\xi)\, U(\xi)\, d\xi \mp \int_0^\infty f(\xi)\, U(-\xi)\, d\xi \,. \tag{6}$$

The second integral which was originally taken from $-\infty$ to 0 has been converted by a change of sign of the variable of integration into an integral from 0 to $+\infty$. The principal solution $U(\xi)$ is then transformed into

$$U(-\xi) = (4\,\pi\, k\, t)^{-\frac{1}{2}} \exp\left\{-\frac{(x+\xi)^2}{4\,k\,t}\right\} \tag{7}$$

which is the expression for a unit heat pole at $x = -\xi$, $t = 0$. Equation (6) becomes:

$$u(x, t) = \int_0^\infty f(\xi)\, G\, d\xi, \qquad G = U(\xi) \mp U(-\xi)\,. \tag{8}$$

This *Green's function* G satisfies all the conditions of p. 61. It has only one heat pole in the region $0 < x < \infty$, since the additional heat pole at $x = -\xi$ lies outside the region; it also complies with the condition that G satisfy the *adjoint* equation in the variables ξ, τ, since in our case G is independent of τ and a change of sign in τ becomes immaterial.

It would be more intuitive to start from a single heat pole at $x = \xi$ and to reflect it on the boundary $x = 0$, with a negative or positive sign of U depending on whether we impose condition (3a) or (3b). In this manner we would first construct Green's function and then reconstruct the given initial temperature $f(x)$ by the successive superposition of the heat poles of strength $f(\xi)\, d\xi$. From now on we shall use mainly this intuitive process, that is we restrict ourselves to the *construction of Green's function* from which we can write down the solution for arbitrary initial temperature $f(x)$ as in (8). We first use this

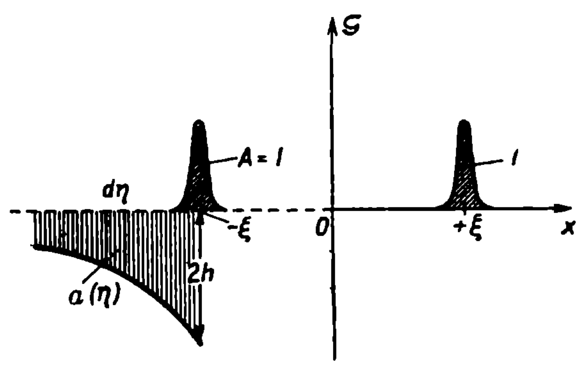


Fig. 15. Green's function for a linear heat conductor which is bounded on one side with outer heat conduction. A heat pole is at $x = +\xi$, its mirror image at $x = -\xi$, with an associated continuous spectrum of heat sources.

process for the somewhat more complicated boundary condition (3c). (In problem III.1 we shall treat the same boundary condition according to Fourier's process.)

We notice at once that an isolated reflected point $x = -\xi$ will not be sufficient, but that we also need a continuous sequence of heat sources which we shall place at all points $\eta < -\xi$. Let A and $a(\eta)\, d\eta$ be the yields of the isolated and the continuous heat sources (see Fig. 15). The corresponding function G is then

$$G = U(\xi) + AU(-\xi) + \int_{-\infty}^{-\xi} a(\eta)\, U(\eta)\, d\eta \tag{9}$$
$$= (4\pi k t)^{-\frac{1}{2}} \left[e^{-\frac{(x-\xi)^2}{4kt}} + A\, e^{-\frac{(x+\xi)^2}{4kt}} + \int_{-\infty}^{-\xi} a(\eta)\, e^{-\frac{(x-\eta)^2}{4kt}}\, d\eta \right]$$

Hence at $x = 0$:

$$(4\pi k t)^{\frac{1}{2}}\, G = (1 + A)\, e^{-\frac{\xi^2}{4kt}} + \int_{-\infty}^{-\xi} a(\eta)\, e^{-\frac{\eta^2}{4kt}}\, d\eta. \tag{10}$$

From (9) we form $\partial G/\partial x$. Then if we replace $\partial/\partial x$ by $-\partial/\partial\eta$ under the integral sign we obtain for $x = 0$

$$(4\pi k t)^{\frac{1}{2}} \frac{\partial G}{\partial x} = \xi\left(\frac{1}{2kt} - \frac{A}{2kt}\right) e^{-\frac{\xi^2}{4kt}} - \int_{-\infty}^{-\xi} a(\eta)\, \frac{\partial}{\partial\eta} e^{-\frac{\eta^2}{4kt}}\, d\eta. \tag{10a}$$

and after integrating by parts:

$$(4\pi k t)^{\frac{1}{2}} \frac{\partial G}{\partial x} = \left[(1 - A)\frac{\xi}{2kt} - a(-\xi)\right] e^{-\frac{\xi^2}{4kt}} + \int_{-\infty}^{-\xi} a'(\eta)\, e^{-\frac{\eta^2}{4kt}}\, d\eta. \tag{11}$$

If we substitute (10) and (11) in condition (3c) with $\partial/\partial n$ replaced by $-\partial/\partial x$, then (3c) must be satisfied identically for all $t > 0$. By setting the terms of different time dependence individually equal to zero we obtain:

$$A - 1 = 0 \quad \ldots \quad A = +1, \tag{12}$$

$$a(-\xi) + h(1 + A) = 0 \quad \ldots \quad a(-\xi) = -2h, \tag{13}$$

and the differential equation:

$$a'(\eta) - h\, a(\eta) = 0 \tag{14}$$

Considering (13) we see that (14) has the solution

$$a(\eta) = b\, e^{h\eta} = -2h\, e^{h(\xi+\eta)}$$

This determines the constant A and the function $a(\eta)$. The fact that this determination is unique will be demonstrated in §17.

The result is

$$(4\pi k t)^{\frac{1}{2}}\, G = e^{-\frac{(x-\xi)^2}{4kt}} + e^{-\frac{(x+\xi)^2}{4kt}} - 2h\, e^{h\xi} \int_{-\infty}^{-\xi} e^{-\frac{(x-\eta)^2}{4kt}}\, e^{h\eta}\, d\eta\,. \tag{15}$$

For numerical applications this integral can be reduced to the tabulated normal error function.[2]

Only when the given initial values $f(x)$ are particularly simple will it be more convenient to use equations (1) and (2) instead of the more intuitive method of Green's function. We illustrate this by an example which also shows the translation of problems of heat conduction into the language of *diffusion problems.*

Let the bottom section of a cylindrical vessel, $0 < x < H$, be filled with a concentrated solution (say $CuSO_4$); above it let there be a layer of pure solvent (water) to an arbitrary height $x = \infty$. Let the concentration of the solution be u and let the initial concentration be 1. At the base of the cylinder we have $\partial u/\partial x = 0$ at all times, since the dissolved salt molecules cannot penetrate the base.

This condition may also be satisfied by extending the vessel downward and by prescribing the reflected initial distribution as above. (For a finite height of water column one would have to use the somewhat more complicated reflection process of §16.) The initial distribution of u is then:

$$f(x) = \begin{cases} 1 \cdots -H < x < +H\,, \\ 0 \cdots \quad H < |x| < \infty\,. \end{cases}$$

Equation (2) yields

$$u(x,t) = \frac{1}{\sqrt{\pi}} \int_{z_1}^{z_2} e^{-z^2}\, dz\,. \tag{16}$$

[2] E.g., in Jahnke-Emde's tables of functions, 3rd ed., Teubner, Leipzig, 1938, p. 24

The values for the limits of integration z_1 and z_2 are obtained if in (1a) we let $\xi = \pm H$:

$$z_1 = \frac{-H - x}{\sqrt{4\,kt}}, \qquad z_2 = \frac{H - x}{\sqrt{4\,kt}}. \tag{17}$$

Using the customary notation

$$\Phi(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-z^2}\, dz \tag{18}$$

for the error function, we can write our solution (16) with astonishing simplicity:

$$u(x, t) = \frac{1}{2}\{\Phi(z_2) - \Phi(z_1)\}. \tag{19}$$

§14. The Problem of the Earth's Temperature

We treat the surface of the earth as a plane and assume an averaged *purely periodic* temperature $f(t)$ (annual averaged or daily averaged temperature). In order to determine the temperature in the earth's interior[3] we can in general use the method described in Fig. 13, by setting $x_0 = 0$ (surface of the earth), $x_1 = \infty$ (great depth), and $u_0 = f(t)$ for $x = 0$. It is convenient in our case to expand $f(t)$ into a complex Fourier-series:

$$f(t) = \sum_{n=-\infty}^{+\infty} C_n e^{2\pi int/T}, \qquad T = \text{length of year or day} \tag{1}$$

and to set for the temperature in the interior of the earth at a depth x:

$$u(x, t) = \sum_{n=-\infty}^{+\infty} C_n u_n(x)\, e^{2\pi int/T}. \tag{2}$$

Each individual term of this series must satisfy the basic law of heat conduction. This yields the ordinary differential equation for u_n:

$$\frac{d^2 u_n}{dx^2} = p_n^2 u_n \quad \text{with} \quad p_n^2 = \frac{2\pi in}{kT}. \tag{3}$$

In order that (2) go into (1) for $x = 0$, we must have (3a)

$$u_n(0) = 1. \tag{3a}$$

[3] The physical problem of "geothermic depth" (increase of temperature in the interior of the earth due to radioactive or nuclear processes) is of course ignored here.

Depending on whether n is positive or negative we set

$$2\,in = (1 \pm i)^2\,|n|$$

and

(4) $$p_n = (1 \pm i)\,q_n\,, \qquad q_n = \sqrt{\frac{|n|\,\pi}{kT}} > 0.$$

The general solution of (3) is then

(5) $$u_n(x) = A_n\,e^{(1\pm i)q_n x} + B_n\,e^{-(1\pm i)q_n x}.$$

Here we must have $A_n = 0$, since otherwise the temperature would become infinite for $x \to \infty$, and $B_n = 1$, to satisfy (3a). Substituting this in (2) we get

(6) $$u(x,t) = \sum_{n=-\infty}^{+\infty} C_n\,e^{-(1\pm i)q_n x}\,e^{2\pi i n t/T}\,.$$

In order to translate this into real language we write for $n > 0$

$$C_n = |C_n|\,e^{i\gamma_n}\,.$$

According to (1.13) C_n for $n < 0$ has the same absolute value but the negative phase. Equation (6) then becomes

(7) $$u(x,t) = C_0 + 2\sum_{n=1}^{\infty} |C_n|e^{-q_n x}\cos\left(2\pi n\,\frac{t}{T} + \gamma_n - q_n x\right).$$

We see that the amplitude $|C_n|$ of the n-th partial wave is *damped exponentially* with increasing depth x, and that this damping increases with increasing n. At the same time the phase of the partial wave is *retarded* increasingly with increasing x and n.

We now consider the numerical values. For an average type of soil we have the approximate temperature conductivity

$$k = 2\cdot 10^{-3}\,\frac{\mathrm{cm}^2}{\mathrm{sec}}.$$

For the period of one year $T = 365 \times 24 \times 60 \times 60 = 3.15 \times 10^7$ sec. and for $x = 1\,m = 100$ cm. we have then

(8) $$q_1 x = 0.7 \sim \frac{\pi}{4}\,, \qquad e^{-q_1 x} \sim \frac{1}{2}\,.$$

At a depth of 4 meters we already have a "phase lag" $q_1 x = \pi$ and an amplitude damping, $2^{-4} = 1/_{16}$. Even for the first and principal partial wave of temperature fluctuation *it is winter at a depth of 4 meters when it is summer on the surface; the amplitude is only a fraction of the surface amplitude.* For the higher partial waves $n > 1$ the phase lag and amplitude damping are correspondingly greater owing to the factor $\sqrt{|n|}$ in q_n. We may say that the earth acts as a *harmonic analyzer* (see p. 4) by singling out the principal (though much weakened) wave from among all partial waves.

As a special example we consider the yearly curve of an "extremely continental climate," namely a uniform summer temperature and the same negative winter temperature which we shall set arbitrarily $= \pm 1$. This year-curve is represented graphically by the meander line of Fig. 1. and analytically by (2.2)

$$(9) \quad u(0,t) = \frac{4}{\pi}\left(\sin\tau + \frac{1}{3}\sin 3\tau + \frac{1}{5}\sin 5\tau + \cdots\right), \qquad \tau = 2\pi\frac{t}{T}.$$

In order to obtain the corresponding series $u(x,t)$ we must, according to (2.1a), specialize the coefficients C in (7) as follows:

$$C_{2n} = 0, \quad |C_{2n+1}| = \frac{2}{\pi(2n+1)}, \qquad \gamma_{2n+1} = -\frac{\pi}{2}.$$

However it is somewhat simpler to apply the calculation process used for (2) directly to equation (9). We immediately get:

$$(9a) \quad u(x,t) = \frac{4}{\pi}\left(e^{-q_1 x}\sin(\tau - q_1 x) + \frac{1}{3}e^{-q_3 x}\sin(3\tau - q_3 x) + \cdots\right)$$

Then, substituting the values of the q we get for $x = 100$ cm.

$$(9b) \quad \begin{aligned} u(x,t) \approx \frac{4}{\pi}\Big[\frac{1}{2}\sin\left(\tau - \frac{\pi}{4}\right) &+ \frac{1}{3\cdot 3.4}\sin\left(3\tau - \frac{\sqrt{3}\pi}{4}\right) \\ &+ \frac{1}{5\cdot 4.8}\sin\left(5\tau - \frac{\sqrt{5}\pi}{4}\right) + \cdots\Big] \end{aligned}$$

and for $x = 400$ cm.

$$(9c) \quad \begin{aligned} u(x,t) \approx \frac{4}{\pi}\Big(\frac{1}{16}\sin(\tau - \pi) &+ \frac{10^{-2}}{3\cdot 1.3}\sin(3\tau - \sqrt{3}\pi) \\ &+ \frac{10^{-3}}{5\cdot 5.3}\sin(5\tau - \sqrt{5}\pi) + \cdots\Big). \end{aligned}$$

A comparison of (9) with (9b,c) shows clearly the influence of depth on the amplitude and the phase of the temperature process.

This shows the usefulness of a deep cellar. It has not only much smaller temperature fluctuations than the surface of the earth, but is also warmer in winter than in summer (or it would be if there were no air flow).

Our conclusions become even more striking if we pass from the consideration of an averaged yearly temperature to that of an averaged *daily* temperature. The q_n are then increased by the factor $\sqrt{365} \sim 19$. Hence the damping and phase lag which for a yearly period belongs to a depth x occurs now at a depth of $x/19$. The decrease of amplitude to 1/16 for the principal term (see (9c)) and the reversal of the time of day (midnight instead of noon) will now occur at a depth of only $x = 400/19 = 21$ cm. Hence the daily fluctuations of temperature enter into the earth with noticeable intensity for only a few centimeters; the whole process takes place in a *thin surface layer*.

We deal here with an obvious analogue to the *skin effect* of electricity. The fact that in practice it is particularly observable on cylindrical wires makes no difference here; for a conductor bounded by a plane it occurs quantitatively in almost the same manner. Our daily curve corresponds to an alternating current of high frequency, our yearly curve to one that is 365 times slower. We know from §7 that the differential equations are the same in both cases, but for electricity we interpret the coefficient k as the specific resistance of the conductor.

§15. The Problem of a Ring-Shaped Heat Conductor

We now turn to the case of a heat conductor of finite length 1. However, at its two ends $x = \pm {}^1/_2$ we do not prescribe the boundary conditions a),b) or c) of p. 63, but instead the much simpler *condition of periodicity*. By this we mean that not only u but also all its derivatives

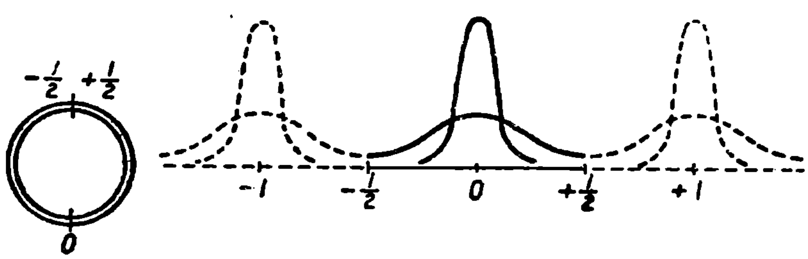


Fig. 16. Heat conduction in a ring. Heat pole at $x = 0$ with periodic repetitions. Temperature distribution for $kt < 1$ (steep curve) and for $kt > 1$ (flat curve).

shall coincide at the ends. We achieve this by bending our rod into a ring so that the two ends coincide. The shape of the ring is of no importance since, as in all cases of linear heat conduction, we must consider the lateral surface of the ring as adiabatically closed. In Fig. 16 we have drawn a circular ring.

As initial temperature we take $f(x)$ which is arbitrary but symmetric with respect to $x = 0$. Its Fourier expansion is a pure cosine series which automatically satisfies the periodicity condition at the ends From (4.1) and (4.2) we get, after setting $a = 1/2$:

$$(1) \qquad f(x) = \sum_{n=0}^{\infty} A_n \cos(2\pi n x), \qquad \begin{aligned} A_0 &= \int_{-1/2}^{+1/2} f(z)\, dz, \\ A_n &= 2\int_{-1/2}^{+1/2} f(z) \cos(2\pi n z)\, dz. \end{aligned}$$

In order to obtain the corresponding solution $u(x,t)$ of the equation of heat conduction we merely must multiply the n-th term by

$$e^{-(2\pi n)^2 k t}$$

We then obtain

$$(2) \qquad u(x, t) = \sum_{n=0}^{\infty} A_n e^{-4\pi^2 n^2 k t} \cos(2\pi n x).$$

We now consider $f(x)$ to be a " δ function" by writing

$$f(x) = 0 \quad \text{for} \quad x \neq 0, \quad \text{but} \quad \int_{-\varepsilon}^{+\varepsilon} f(x)\, dx = 1.$$

Then in (1) we get $A_0 = 1$, $A_1 = A_2 = \ldots = 2$, and if we replace u by the customary ϑ we get:

$$(3) \qquad \vartheta(x, t) = 1 + 2\sum_{n=1}^{\infty} e^{-4\pi^2 n^2 k t} \cos(2\pi n x).$$

The letter ϑ stands for the *theta-function* which was introduced by C. G. J. Jacoby in the theory of elliptic functions and which is of paramount importance in all numerical computations.[4] The fact that it satisfies the equation of heat conduction is frequently used there as an incidental property, whereas we use this property for the definition of ϑ

We now have to adjust our notation t to the theory of the ϑ function by setting

[4] The reason for its special convergence was mentioned in §3, p. 15: since the ϑ series together with all its derivatives is periodic, and therefore has no jumps at $x = +1/2$ and $x = -1/2$, its terms vanish with increasing n more rapidly than any power of n.

$$\tau = 4 \pi i k t. \tag{4}$$

This τ, which of course has nothing to do with the symbol $t - \tau$ of the principal solution U, does not have the dimension of time and is positive-imaginary in our case. (In the theory of elliptic function τ is in general complex with positive imaginary part, namely the ratio of the two periods of these functions). Written in terms of τ (3) becomes:

$$\vartheta(x|\tau) = 1 + 2 \sum_{n=1}^{\infty} e^{i\pi\tau n^2} \cos(2 \pi n x). \tag{5}$$

This form converges very well for large τ or, in other words, for *large kt*. It represents the *later phases* of the damping out of the unit source exceptionally well, but it does not help us for the beginning of this process. We therefore complete Fourier's process which, in analogy to the reflection process, is based on a periodic repetition of the initial state (see the right half of Fig. 16).

We have rolled off the cut ring on the x-axis both to the right and the left in an infinite sequence. From the heat source $U_0(x,t)$ given in the ring we get, at the points $x = n$ $(n = \pm 1, \pm 2, \ldots)$, the identical heat sources:

$$U_n(x, t) = \frac{1}{\sqrt{4 \pi k t}} \exp\left\{- \frac{(x-n)^2}{4 k t}\right\}. \tag{6}$$

In the series

$$u(x, t) = \sum_{n=-\infty}^{+\infty} U_n(x, t) \tag{7}$$

we have a second representation of the damping out process which converges excellently for *small values of kt*. This is so because for such values we need consider only U_0 and its immediate successors, the subsequent U_n having no effect on account of the factor $\exp(n^2/4 k t)$ of (6). Equation (7) is therefore the desired complement of (5). The figure shows the nature of both representations: the flat curve shows the behavior for large kt according to (5), the steep curve shows the behavior for small kt according to (7).

Oddly enough we can bring (7) into a form very similar to that of the ϑ series in (5). All we must do is put the factor $\exp(-x^2/4 k t)$ outside the summation and combine the terms with $\pm n$. Equation (7) then becomes

$$u(x, t) = (4 \pi k t)^{-\frac{1}{2}} \exp\left\{- \frac{x^2}{4 k t}\right\} \cdot \left[1 + 2 \sum_{n=1}^{\infty} e^{-\frac{n^2}{4kt}} \cos \frac{i n x}{2 k t}\right]. \tag{7a}$$

If we replace t by τ according to (4) then the bracketed term becomes

$$1+2\sum_{n=1}^{\infty} e^{-\frac{\pi i n^2}{\tau}} \cos 2\pi n \frac{x}{\tau}.$$

This differs from (5) only in that x/τ has replaced x in the argument of the cosines and that $-1/\tau$ has replaced τ in the exponents. Hence the bracket of (7a) is

$$\vartheta\left(\frac{x}{\tau}\middle| -\frac{1}{\tau}\right).$$

Substituting this in (7) and remembering that both (5) and (7) are solutions of the same problem of heat conduction we obtain

$$\vartheta(x|\tau) = \left(\frac{\tau}{i}\right)^{-\frac{1}{2}} \exp\left\{-\frac{\pi i x^2}{\tau}\right\}\cdot\vartheta\left(\frac{x}{\tau}\middle| -\frac{1}{\tau}\right)$$

or conversely

$$\vartheta\left(\frac{x}{\tau}\middle| -\frac{1}{\tau}\right) = \sqrt{\frac{\tau}{i}}\exp\left\{\frac{\pi i x^2}{\tau}\right\}\cdot\vartheta(x|\tau). \tag{8}$$

This is the famous *transformation formula of the* ϑ *function.* It is used in the theory of elliptic functions to transform the series $\vartheta(x|\tau)$ which converges slowly for small τ into the very rapidly converging series $\vartheta\left(\frac{x}{\tau}\middle|-\frac{1}{\tau}\right)$. For us it constitutes the passage from Fourier's method to the method of heat poles. In quantum theory formula (8) is of importance for the rotational energy of diatomic molecules and for the calculation of their specific heat for low temperatures.

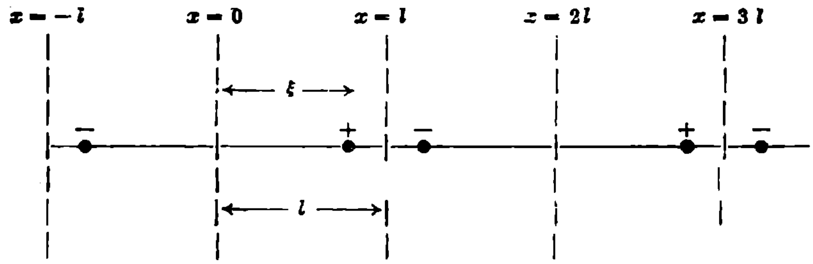
§16. Linear Heat Conductors Bounded on Both Ends

By setting the length of the ring in the preceding section equal to 1, we tacitly introduced a new dimensionless coordinate $x' = x/l$ and wrote x instead of x'. For the case of a rod of length l, which we shall consider now, we must replace x by x/l whenever we apply one of the preceding formulas. The meaning of τ, which has the dimension of x^2, must then be amended in the manner described on the following page.

We first give a table of the problems corresponding to the boundary conditions a) and b) of p. 63 and of their solutions by both Fourier's method and the method of heat poles. The latter leads to an *infinite* sequence of reflections since not only the primary heat pole but also all its images are reflected at both ends of the rod. Let us consider a room with parallel mirrors as an optical example; the chandelier will be reflected in both mirrors not once but in infinite repetition.

a) a)

$u = 0$ for $\begin{cases} x = 0 \\ x = l \end{cases}$

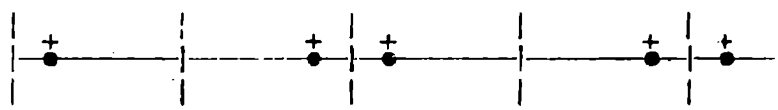


$$f(x) = \sum B_n \sin \pi n \frac{x}{l}, \quad B_n = \frac{2}{l} \int_0^l f(x) \sin \pi n \frac{x}{l} \, d x,$$

$$G = \vartheta\left(\frac{x-\xi}{2 l} \,|\, \tau\right) - \vartheta\left(\frac{x+\xi}{2 l} \,|\, \tau\right).$$

b) b)

$\frac{\partial u}{\partial x} = 0$ for $\begin{cases} x = 0 \\ x = l \end{cases}$



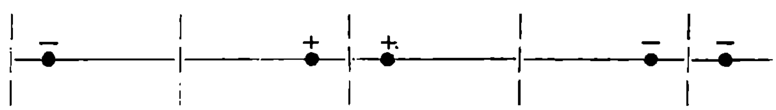
$$f(x) = \sum A_n \cos \pi n \frac{x}{l}, \quad A_n = \frac{2}{l} \int_0^l f(x) \cos \pi n \frac{x}{l} \, d x, \quad A_0 = \frac{1}{l} \int_0^l f(x) \, dx,$$

$$G = \vartheta\left(\frac{x-\xi}{2 l} \,|\, \tau\right) + \vartheta\left(\frac{x+\xi}{2 l} \,|\, \tau\right).$$

a) b)

$u = 0$ for $x = 0$

$\frac{\partial u}{\partial x} = 0$,, $x = l$



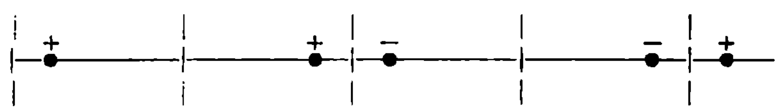
$$f(x) = \sum B_n \sin \pi (n + \tfrac{1}{2}) \frac{x}{l}, \quad B_n = \frac{2}{l} \int_0^l f(x) \sin (n + \tfrac{1}{2}) \frac{x}{l} \, d x,$$

$$G = \vartheta\left(\frac{x-\xi}{4 l} \,|\, \tau\right) - \vartheta\left(\frac{x+\xi}{4 l} \,|\, \tau\right) + \vartheta\left(\frac{x+\xi-2l}{4 l} \,|\, \tau\right) - \vartheta\left(\frac{x-\xi-2l}{4 l} \,|\, \tau\right).$$

b) a)

$\frac{\partial u}{\partial x} = 0$ for $x = 0$

$u = 0$,, $x = l$



$$f(x) = \sum A_n \cos \pi (n + \tfrac{1}{2}) \frac{x}{l}, \quad A_n = \frac{2}{l} \int_0^l f(x) \cos (n + \tfrac{1}{2}) \frac{x}{l} \, d x,$$

$$G = \vartheta\left(\frac{x-\xi}{4 l} \,|\, \tau\right) + \vartheta\left(\frac{x+\xi}{4 l} \,|\, \tau\right) - \vartheta\left(\frac{x+\xi-2 l}{4 l} \,|\, \tau\right) - \vartheta\left(\frac{x-\xi-2l}{4 l} \,|\, \tau\right).$$

We see immediately that the functions $f(x)$ in this table satisfy the boundary conditions a)a) to b)b); these boundary conditions then hold for the corresponding solutions of the boundary value problems $u(x,t)$, which are obtained from $f(x)$ according to *Fourier's method* by multiplying the series of f termwise by

$$e^{-(\pi n/l)^2 kt} \quad \text{or} \quad e^{-[\pi(n+1/2)/l]^2 kt}$$

The diagrams show the positions and the signs of the heat poles according to the *reflection method*. In the first two cases the heat poles are seen to have the period $2l$, in the last two they have the period $4l$. Their summation yields Green's function $G = \Sigma U$ which is expressed here in terms of the ϑ function of the preceding section. In the formulas of the preceding section, where the period was taken equal to 1 and the heat pole was at $x = 0$, we have to replace x by $\frac{x-\xi_i}{2l}$ for a)a) and b)b) and by $\frac{x-\xi_i}{4l}$ for a)b) and b)a), where ξ_i stands for the position of any heat pole of the sequence which is summed by ϑ. (Due to the periodicity the choice of the heat pole is immaterial.) In our formulas we have chosen for ξ_i the heat pole of the initial region $0 < x < l$ or of one of the adjacent regions. From Green's function we get the solution of the boundary value problem for arbitrary initial values $u(x,0) = f(x)$ according to the general formula

$$u(x, t) = \int_0^l f(\xi)\, G(x, \xi; t)\, d\xi \,. \tag{1}$$

We now wish to treat the *boundary condition* c) of p. 64, where we particularly consider the combination a)c). In order to satisfy condition a) at $x = 0$ we set

$$f(x) = \sum_{n=1}^{\infty} B_n \sin \lambda_n \pi \frac{x}{l}. \tag{2}$$

That is to say, in the solution for a)a) we replace the sequence of integers n by the sequence

$$\lambda_1, \lambda_2, \ldots, \lambda_n, \ldots$$

which we wish to determine in such a way that for $x = l$ condition c) is satisfied. This leads to the transcendental equation

$$\lambda_n \frac{\pi}{l} \cos \lambda_n \pi + h \sin \lambda_n \pi = 0 \,.$$

or

(3) $$\tan \lambda_n \pi = -\frac{\pi \lambda_n}{h l}.$$

This is equation (6.2a) with $\alpha = -\pi/h l$; its solution was illustrated in Fig. 7. Hence we are dealing here with a typical case of *anharmonic Fourier analysis.* The values of the coefficients B_n in (2) can be taken directly from (6.3b). We obtain the final solution of our boundary value problem a)c) if we multiply the terms in the sum of (2) by the required time factors:

(4) $$u(x, t) = \sum_{n=1}^{\infty} B_n \sin \lambda_n \pi \frac{x}{l} \cdot \exp\left\{-\left(\frac{\lambda_n \pi}{l}\right)^2 k t\right\}.$$

We mentioned in §6 that the formal computation of the coefficients B can be replaced by a physically meaningful one. We shall do this now in such a manner that we shall be able to refer to this case in all future expansions in "eigenfunctions."

We consider two arbitrary terms of (2):

(5) $$u_n = \sin \lambda_n \frac{\pi x}{l}, \qquad u_m = \sin \lambda_m \frac{\pi x}{l};$$

They satisfy the differential equations:

(5 a) $$\frac{d^2 u_n}{dx^2} + k_n^2 u_n = 0, \qquad \frac{d^2 u_m}{dx^2} + k_m^2 u_m = 0, \qquad \begin{cases} k_n = \lambda_n \frac{\pi}{l}, \\ k_m = \lambda_m \frac{\pi}{l}. \end{cases}$$

Hence

(5 b) $$u_m \frac{d^2 u_n}{dx^2} - u_n \frac{d^2 u_m}{dx^2} = (k_m^2 - k_n^2)\, u_m u_n.$$

The left side is a total derivative (for the case of Green's theorem, p. 44, we spoke of a "divergence"). The integration of (5b) over the fundamental region $0 < x < l$ reduces to the boundary points on the left (in Green's theorem we said "to a boundary integral"). From this we obtain the value of the right side without further calculation:

(6) $$(k_m^2 - k_n^2) \int_0^l u_m u_n \, dx = u_m \frac{du_n}{dx} - u_n \frac{du_m}{dx} \Big|_{x=0}^{x=l}.$$

Here the right side vanishes for $x = 0$ since according to (5) we have then $u_n = u_m = 0$; but it also vanishes for $x = l$, since the boundary condition c) holds for every individual term of (2) and hence the du/dx are proportional to the u. Therefore the *condition of orthogonality* (6.3) is satisfied for $k_m \neq k_n$.

We shall show in exercise III.2 that the normalizing integral (6.3b) can be obtained almost without further calculation.

The expressions as well as the mathematical formulations in this section were based on the assumption that the lateral surface of the rod is closed to heat flow, an assumption that is open to reasonable doubt. We shall show now that our formulas can be used also in the case of incomplete closure with respect to heat flow.

Instead of imposing the adiabatic condition b) on an element of area $d\sigma$ of the lateral surface of our rod, which we assume to be a circular cylinder, we impose condition c) which states that an amount of heat

$$-\varkappa \frac{\partial u}{\partial n} d\sigma = \varkappa\, h\, u\, d\sigma$$

passes out through $d\sigma$ per unit of time. We apply this to the case of an element of a cylindrical rod of altitude dx and radius of cross section b so that the lateral area is $2\pi b\, dx$ and the outer normal dn is in the direction of the extended radius. The amount of heat passing out of the lateral surface is

$$dQ_1 = \varkappa\, h\, u \cdot 2\pi b\, dx\, dt. \tag{7}$$

The amount of heat flowing out of the bases $x =$ const. and $x + dx$ $=$ const. of this element is

$$dQ_2 = -\varkappa \frac{\partial^2 u}{\partial x^2} \cdot \pi b^2\, dx\, dt \tag{8}$$

According to (7.9) the total outflux of heat from the rod element is equal to the product of div G dt with its volume. Hence we have

$$\text{div } \mathbf{G} \cdot \pi b^2 dx\, dt = dQ_1 + dQ_2. \tag{9}$$

and after substituting (7) and (8)

$$\text{div } \mathbf{G} = \frac{2\varkappa h}{b} u - \varkappa \frac{\partial^2 u}{\partial x^2}. \tag{10}$$

According to (7.11) div G is proportional to $-\partial u/\partial t$. Substituting this and dividing by $\varkappa$ we obtain

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{k}\frac{\partial u}{\partial t} + \frac{2h}{b} u. \tag{11}$$

Hence the "outer-heat conduction" through the lateral surface only modifies our differential equation by the additional term of equation (11). Our derivation depended on the assumption that the linear character of the thermal state, i.e., its sole dependence on x, is not affected by the lateral radiation, an assumption which seems plausible for sufficiently small cross section.

The integration of (11) is very simple. We let:

$$u = v\, e^{-\lambda t};$$

After division by $\exp(-\lambda t)$ equation (11) becomes:

$$\frac{\partial^2 v}{\partial x^2} = \frac{1}{k}\frac{\partial v}{\partial t} + \left(\frac{2h}{b} - \frac{\lambda}{k}\right) v,$$

which is the ordinary equation of heat conduction if we set:

$$\lambda = \frac{2hk}{b}. \tag{12}$$

Hence all the developments of this chapter are valid for a rod with outer heat conduction if we multiply by the factor $\exp\left(-\frac{2hk}{b}t\right)$

In exercises III.3 and III.4 we shall see an elegant experimental determination of the ratios inner to outer heat conduction and heat conduction to electron conduction in metals.

§ 17. Reflection in the Plane and in Space

We finally leave the case of linear heat conduction and turn to spatial regions which are bounded by planes and which can be treated by the simple *reflection method.* The corresponding plane regions bounded by straight lines will be treated in a very similar manner.

The simplest case is that of a *half space* with the boundary conditions $u = 0$ or $\partial u/\partial n = 0$. Since we know the spatial function of a heat pole from equation (12.14), we can write Green's function for the half space directly. If we take the boundary of the half space to be $z = 0$ and the source point to be (ξ, η, ζ) then we have

$$(4\pi k t)^{\frac{3}{2}} G = e^{-\frac{r^2}{4kt}} \mp e^{-\frac{r'^2}{4kt}} \begin{cases} r^2 = (x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2, \\ r'^2 = (x-\xi)^2 + (y-\eta)^2 + (z+\zeta)^2 \end{cases} \tag{1}$$

Since for $z = 0$

$$r^2 = r'^2 \quad \text{and} \quad \frac{\partial r^2}{\partial z} = -\frac{\partial r'^2}{\partial z}$$

we get

$$G = 0 \qquad \text{or} \quad \frac{\partial G}{\partial n} = -\frac{\partial G}{\partial z} = 0 \quad \text{for } z = 0.$$

Even for the boundary condition c) of p. 64 we can transfer the solution (13.15) directly to the spatial case. We have

$$(2) \qquad (4\pi k t)^{\frac{3}{2}} G = e^{-\frac{r^2}{4kt}} + e^{-\frac{r'^2}{4kt}} - 2h e^{h\zeta} e^{-\frac{\varrho^2}{4kt}} \int_{-\infty}^{-\zeta} e^{-\frac{(z-\beta)^2}{4kt}} e^{h\beta} d\beta.$$

with

$$\varrho^2 = (x-\xi)^2 + (y-\eta)^2,$$

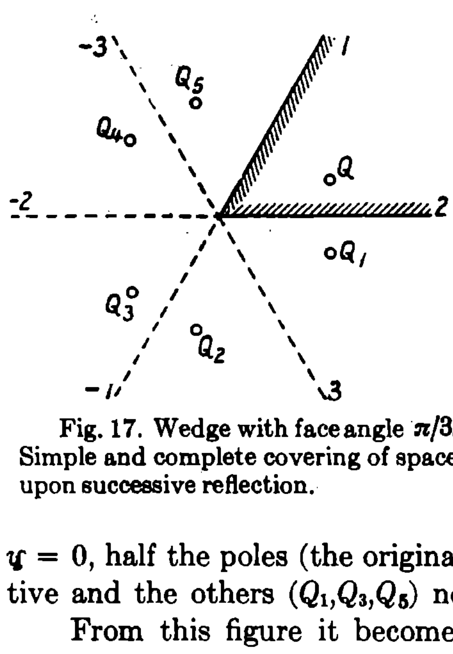


Fig. 17. Wedge with face angle $\pi/3$. Simple and complete covering of space upon successive reflection.

Not all spatial regions bounded by planes can be treated according to the reflection process. It is necessary that, under successive reflections of the original region, space be covered *completely and simply.* We demonstrate this with the example of a *wedge.* If it has an angle of 60° (see Fig. 17) then it is reproduced five times upon successive reflections whereupon the process terminates. Hence Green's function can be represented by a sum of six heat poles where, for the boundary condition $u = 0$, half the poles (the original pole Q and its images Q_2, Q_4) are positive and the others (Q_1, Q_3, Q_5) negative.

From this figure it becomes apparent that the reflection process may be attempted only for those polyhedrons whose *face angles are all submultiples of* π (not merely of 2π). In the case of wedges the angle $2\pi/3 = 120°$ leads to a double covering of space, $3\pi/2$ leads to a triple covering, and every angle which is incommensurable with π leads to an infinite covering. A particularly interesting case is that of space with a half plane removed, a wedge of angle 2π so to speak. Its treatment according to a reflection process requires the study of the principal solution in a "two-sheeted Riemann space" whose branch line is the edge of the half plane.[5]

[5] This solution was given by the author in 1894, *Math. Ann.* **45**, first for the case of heat conduction and soon thereafter for the refraction of light (*ibid.* **47**). For details see Frank-Mises, 2nd ed. (8th ed. of Riemann-Weber), Vieweg, 1935, chapter 20.

Among the polyhedral regions we consider first the interior of a *cube* (the exterior would lead to most complicated ramifications) and as its generalization the *rectangular solid.* The mirror images of the given primary source point form eight superimposed spatial lattices corresponding to the eight combinations of signs $\pm\xi, \pm\eta, \pm\zeta$. Each of these lattices taken separately forms a triply periodic solution of the differential equation, so to speak a higher ϑ function (see below). If the base of a rectangular solid is divided by its diagonals into four isosceles triangles then a rectangular cylinder with one of them as base is also a polyhedron of the required kind. Another example is given by a rectangular cylinder whose base is an equilateral triangle or half an equilateral triangle obtained through bisection by the altitude. A rectangular cylinder whose base is a regular hexagon has face angles of $2\pi/3$ and therefore leads to a double, not a simple, covering of space.

Everything said about the subdivision of rectangular solids is of course true for cubes. In addition, for suitable subdivisions the cube yields permissible tetrahedra: Lamé's "tetrahedra 1/6 and 1/24," of which the former fills out the cube upon six reflections, the latter upon twenty-four, and another tetrahedron which was discovered by Schönflies in his general investigations on crystal structure.[6]

For all these regions we can not only solve the problem of heat conduction but *any physical process of isotropic symmetry*, such as an acoustic, optic, or electric process, by the reflection method. The very word "reflection" reminds us of the optical application.

The set of permissible regions is extended very considerably if we no longer impose *boundary conditions*, but require *periodicity*, as in the case of the ring in the beginning of §15. Then instead of a rectangular solid we can treat an *arbitrary parallelipiped;* all we have to do is to repeat periodically the pole of Green's function in the initial domain in all its images under the translation group. The elliptic ϑ function of the ring is then replaced by a higher ϑ function (hyperelliptic Abelian); however we shall not go into this since there are no immediate physical applications.

Everything said here about spatial regions can be transferred directly to plane regions. Half space is replaced by half plane, rectangular solid by rectangle, the rectangular cylinder whose base is an equilateral triangle by that triangle. In formula (1) for Green's function of half space we have to replace the exponent $\frac{3}{2}$ on the left by the exponent 1,

[6] G. Lamé, Leçons sur la theorie de la chaleur, Paris 1861; he does not use the reflection method but Fourier's method with a suitable continuation of the arbitrary initial distribution. For A. Schönflies' tetrahedron see *Math. Ann.* **34**.

and the three-dimensional square of distances on the right by the corresponding square of distances in the plane.

Unfortunately this method of reflection for problems of heat conduction can not be applied to spherical (or circular) regions (see §23).

§ 18. Uniqueness of Solution for Arbitrarily Shaped Heat Conductors

The physicist may consider such a proof superfluous; we shall consider it, however, on account of its mathematical elegance and the importance of its method.

It will suffice to use Green's theorem of potential theory which we formulated in exercise II.2 as the "second form." The parabolic character of the equation of heat conduction plays no particular role here; it would become important if, as in Fig. 13, we were to impose time dependent boundary conditions, but we shall restrict ourselves here to the boundary conditions a),b),c) of p. 63.

Our heat conductor may have an arbitrary boundary; as part of this boundary we include the boundaries of any possible inner cavities. On this total boundary surface σ there may be given an arbitrary combination of the boundary conditions

$$\text{a) } u = f_1(\sigma), \qquad \text{b) } \frac{\partial u}{\partial n} = f_2(\sigma), \qquad \text{c) } \frac{\partial u}{\partial n} + hu = f_3(\sigma)$$

("non-homogeneous" boundary conditions where f_1, f_2, f_3 are arbitrary point functions on σ, in contrast to the previous "homogeneous" boundary conditions where the right sides are zero). In addition we assume the initial temperature u to be given as an arbitrary point function $f(x,y,z)$.

Let u_1 and u_2 be two different solutions of the equation of heat conduction under these initial and boundary conditions. Their difference $u_1 - u_2 = w$ then satisfies same differential equation as u_1, u_2

$$\Delta w = \frac{1}{k} \frac{\partial w}{\partial t} \tag{1}$$

with a distribution over σ of the "homogeneous" boundary conditions

$$\text{a) } w = 0, \qquad \text{b) } \frac{\partial w}{\partial n} = 0, \qquad \text{c) } \frac{\partial w}{\partial n} + hw = 0 \tag{2}$$

and the initial condition

$$w = 0 \qquad \text{for} \qquad t = 0. \tag{3}$$

Setting both u and v equal to w in Green's theorem of exercise II.2 we get:

$$\int w\,\Delta w\,d\tau + \int (\text{grad } w, \text{grad } w)\,d\tau = \int w\,\frac{\partial w}{\partial n}\,d\sigma. \tag{4}$$

Considering (1) and (2) this becomes

$$\frac{1}{2\,k}\,\frac{\partial}{\partial t}\int w^2\,d\tau = -\int D w\,d\tau - \int h w^2\,d\sigma_c. \tag{5}$$

where Dw is the so-called first differential parameter:

$$Dw = \left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2.$$

The last term of (5) is to be considered only over that part of σ on which the boundary condition c) holds, as indicated by the subscript c attached to $d\sigma$.

Equation (5) contains a contradiction: the right side is *negative* since $h > 0$ as established in (13.5). (We no longer have to assume that h is a constant; h may vary on the surface σ_0 depending on the local structure of the surface.) The left side of (5) is certainly *positive* for small t, since w^2 is 0 for $t = 0$ and therefore can only increase for increasing t. In order to make this contradiction even more apparent and to extend it to arbitrary t, we integrate (5) with respect to t:

$$\frac{1}{2\,k}\int w^2\,d\tau = -\int_0^t dt \int D w\,d\tau - \int_0^t dt \int h\,w^2\,d\sigma_c. \tag{6}$$

The only possibility of removing this contradiction is in setting:

$$w = 0, \qquad \text{hence} \qquad u_1 = u\,. \tag{7}$$

This *uniqueness result* can also be expressed as follows: in heat conduction there exist no eigenfunctions for any shape of the conductor (see Chapter V). In this sense heat conduction and all analogous *equalization processes* differ in a characteristic manner from *oscillation processes*.

CHAPTER IV

Cylinder and Sphere Problems

This chapter serves to complete our stock of mathematical tools rather than to solve any new physical problems. A necessary part of the tools of a mathematical physicist are cylindrical and spherical harmonics. We shall develop these tools with the help of simple physical considerations rather than in an abstract mathematical manner. We shall connect spherical harmonics with *potential theory* (in which they first arose) and cylindrical harmonics with the *wave equation* and its simplest solution, the monochromatic wave.

§ 19. Bessel and Hankel Functions

We assume the time dependence in the wave equation (7.4) to be periodic, and write it conveniently in the form

(1) $$e^{-i\omega t}, \qquad \omega = \text{circular frequency}.$$

We introduce

(2) $$k = \frac{\omega}{c}, \qquad k = \text{wave number};$$

and then write (7.4) for one and two dimensions:

(3a) $$\frac{d^2u}{dx^2} + k^2 u = 0, \qquad \text{(3b)} \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + k^2 u = 0.$$

Equation (3a) has the integrals

(4a) $$u = A\, e^{ikx} \quad \text{and} \quad u = B\, e^{-ikx}.$$

Because of our choice of negative sign in (1) the first equation stands for a plane wave which progresses in the direction of the positive x-axis, the second for one which progresses in the direction of the negative x-axis. The fact that it is simpler to operate with a wave which progresses in the positive x-direction is the main reason for the choice of sign in (1). For the two-dimensional case (3b) we get

(4b) $$u = A\, e^{i(ax+by)}, \qquad a^2 + b^2 = k^2, \qquad \begin{cases} a = k\cos\alpha, \\ b = k\sin\alpha. \end{cases}$$

Introducing the plane polar coordinates r, φ with

$$x = r\cos\varphi, \qquad y = r\sin\varphi,$$

we get from (4b)

$$u = A\,e^{ikr\cos(\varphi-\alpha)}. \tag{5}$$

Equation (5) represents a plane wave which progresses in the direction $\varphi = \alpha$; for $\alpha = 0$ it becomes (4a). From such solutions we can construct the general solution of (3b) by summation (integration) over with coefficients A which may depend on α.

Written in terms of r,φ equation (3b) reads

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \varphi^2} + k^2 u = 0 \tag{6}$$

or if we set $\varrho = kr$,

$$\frac{\partial^2 u}{\partial \varrho^2} + \frac{1}{\varrho}\frac{\partial u}{\partial \varrho} + \frac{1}{\varrho^2}\frac{\partial^2 u}{\partial \varphi^2} + u = 0. \tag{6a}$$

We seek the solutions of this equation which have the form

$$u = Z_n(\varrho)\,e^{in\varphi}, \tag{7}$$

For this purpose we set

$$A = c_n e^{in\alpha},\ c_n \text{ being a constant independent of } \alpha,$$

and integrate with respect to α between suitable limits β and γ:

$$u = c_n \int_\beta^\gamma e^{i\varrho\cos(\varphi-\alpha)} e^{in\alpha}\,d\alpha. \tag{8}$$

Equation (8), unlike (5), does not represent *one* wave of direction α, but a *bundle* of waves with directions varying from $\alpha = \beta$ to $\alpha = \gamma$, which obviously satisfies the differential equation (6a). In order to bring (8) into the form (7) we write

$$\alpha = w + \varphi, \qquad w_0 = \beta - \varphi, \qquad w_1 = \gamma - \varphi, \tag{8a}$$

Equation (8) then becomes

$$u = c_n \int_{w_0}^{w_1} e^{i\varrho\cos w} e^{inw}\,dw \cdot e^{in\varphi}. \tag{9}$$

The coefficient here of $e^{in\varphi}$ is a function of ϱ alone if, and only if, we remove the dependence of w_0 and w_1 on φ. This is done in (8a) by letting β and γ, and with them w_0, w_1, increase to infinity in some way. In order to accomplish this we first must investigate the convergence of the integral in (9) in the neighborhood of infinity (see Fig. 18). This is obviously a question of determining those regions of the complex w-plane in which the real part of the exponent $i\varrho \cos w$ of (9) *is negative.* We assume for the time being that ϱ is real and positive and set

$$w = p + iq, \text{ and hence } \operatorname{Re}\{i \cos w\} = \sinh q \sin p.$$

Hence for the upper half of the w-plane, $q > 0$, we have

$$(10\,a) \qquad \sin p < 0, \qquad -\pi < p < 0 \mod 2\pi \;{}^{1}$$

and for the lower half of the w-plane, $q < 0$,

$$(10\,b) \qquad \sin p > 0, \qquad 0 < p < \pi \mod 2\pi$$

Since conditions (10a,b) depend on only the real part p of w and not on q, we know that the regions in question are strips which are parallel to the imaginary axis. The regions for which the passage of w_0 and w_1 to infinity is permissible are shaded in Fig. 18.

If ϱ is not real and positive, say $= |\varrho| e^{i\Theta}$, then the above pattern is maintained and is only shifted by $\pm\Theta$ in the direction of the real axis, where the + and − signs are for the upper and lower half planes. In the convergence considerations of (10a,b) we merely have to replace $\sin p$ by $\sin(p \mp \Theta)$ (see the beginning of exercise IV.2).

For each choice of the limits w_0, w_1 which satisfies the stated conditions the coefficient of $e^{in\varphi}$ in (9) is a possible form of the *general cylinder function* $Z_n(\varrho)$ of (7). Substituting (7) in (6a) we see that the functions Z_n satisfy the differential equation:

$$(11) \qquad \frac{d^2 Z_n}{d\varrho^2} + \frac{1}{\varrho}\frac{dZ_n}{d\varrho} + \left(1 - \frac{n^2}{\varrho^2}\right) Z_n = 0.$$

A. The Bessel Function and its Integral Representation

Our first special choice is

$$(12) \qquad \begin{aligned} w_0 &= a + i\infty, & -\pi &< a < 0, \\ w_1 &= b + i\infty, & \pi &< b < 2\pi. \end{aligned}$$

The corresponding path of integration is denoted in Fig. 18 by w_0 the function obtained is called a *Bessel function* if the factor c_n in (9) is normalized by:

[1]Two numbers p and p' are said to be "congruent modulo 2π" (written $p \equiv p' \mod 2\pi$), if $p - p'$ is an integral multiple of 2π.

(13) $$c_n = \frac{1}{2\pi} e^{-in\pi/2}.$$

Using the common[2] notation I_n we obtain

(14) $$I_n(\varrho) = \frac{1}{2\pi} \int_{W_0} e^{i\varrho \cos w} e^{in(w-\pi/2)} dw.$$

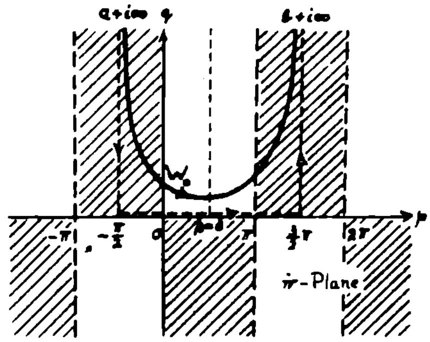


Fig. 18. Regions of the plane $w = p + iq$ in which the real part of $i\varrho \cos w$ is negative are shaded. The path of integration W_0 for the Bessel function I goes from $w_0 = a + i\infty$ to $w_1 = b + i\infty$. In addition to w we use the variable of integration $\beta = w - \pi/2$.

The normalization (13) has been chosen so that $I_0(\varrho) = 1$ for $\varrho = 0$ and $I_n(\varrho)$ is real for arbitrary n and ϱ. The former follows from (14) if we pass to the rectangular form of W_0, which is depicted in Fig. 18 by the dotted path. We thereby cause the two partial integrals along the parts parallel to the imaginary axis (which are otherwise divergent for $\varrho \to 0$) to be complex conjugates. In order to prove this we make the substitution $w - \pi/2 = \beta$ The rectangular path W_0 is then, in terms of β,

$$-\pi + i\infty \to -\pi \to \pi \to \pi + i\infty,$$

which lies symmetric with respect to the β- axis. For real ϱ and n, $I_n(\varrho)$ decomposes into the two parts:

(15) $$I_n(\varrho) = \frac{1}{2\pi} \int_{-\pi}^{+\pi} e^{i(n\beta - \varrho \sin \beta)} d\beta - \frac{\sin n\pi}{\pi} \int_0^{\infty} e^{-(n\gamma + \varrho \sinh \gamma)} d\gamma,$$

where the second term is obtained from the integrals over the two paths $\pi \to \pi + i\infty$ and $-\pi + i\infty \to -\pi$ by the substitution $\beta = \pm \pi + i\gamma$; it does not vanish in general as it did for $n = 0$. Hence under the normalization of (14) $I_n(\varrho)$ is indeed real for real ϱ and n.

Since our integral representation converges for all values of ϱ it follows that $I_n(\varrho)$ is an *everywhere regular transcendental function* except for a single *essential singularity* at infinity and a *branch point* of order n at $\varrho = 0$ which for negative n is also a pole of the same order.

If n is an integer then the second term in (15) vanishes and we get

(16) $$I_n(\varrho) = \frac{1}{2\pi} \int_{-\pi}^{+\pi} e^{i(n\beta - \varrho \sin \beta)} d\beta.$$

[2] In the English literature one writes J_n instead of I_n and sets $I_n(\rho) = J_n(i\rho)$. We wish to reserve the letter J to denote "intensity" and we shall need no special symbol for $I_n(i\rho)$.

If we express the exponential function in terms of trigonometric functions and consider the odd and even character of the sine and cosine, then we get a representation which was given by Bessel:

$$(17) \qquad I_n(\varrho) = \frac{1}{\pi} \int_0^\pi \cos(\varrho \sin\beta - n\beta)\, d\beta .$$

We can see this directly from our original integral with respect to w. In the rectangular path W_0 the two parts which are parallel to the imaginary axis will cancel for integral values of n, and only the section of the real axis from $-\pi/2$ to $3\pi/2$ remains. Due to the periodicity of the integrand this can be replaced by the path from $-\pi$ to $+\pi$. We thus obtain

$$(18) \qquad I_n(\varrho) = \frac{1}{2\pi} \int_{-\pi}^{+\pi} e^{i\varrho \cos w}\, e^{in(w-\pi/2)} dw ,$$

which agrees with (16). The integral over a complex path W_0 as in (14) has a great advantage over the real representations in that it is not limited to integral values of n but remains valid for arbitrary n. The integral (14) is first mentioned in Schläfli 1871,[3] though only with a rectangular path of integration. The following integrals (22) were first published by the author in 1896.

Since the differential equation (11) depends only on n^2 we know that if I_n is a solution then so is I_{-n}. The general solution can therefore be written in the form

$$(19) \qquad Z_n(\varrho) = c_1 I_n(\varrho) + c_2 I_{-n}(\varrho)$$

However this holds only for *non-integral* n. For *integral* n, I_n and I_{-n} are not linearly independent; we have rather

$$(19a) \qquad I_{-n}(\varrho) = (-1)^n I_n(\varrho).$$

This follows directly from (16) if in $I_{-n}(\varrho)$ we make the substitution $\beta = \pi - \beta'$.

B. The Hankel Function and its Integral Representation

As limits of integration in (9) we now choose

$$(20) \qquad \begin{aligned} w_0 &= a_1 + i\infty, \quad && -\pi < a_1 < 0, \\ w_1 &= b_1 - i\infty, \quad && 0 < b_1 < \pi; \end{aligned}$$

[3] For details see G. N. Watson, A Treatise on the Theory of Bessel Functions, Cambridge 1922, p. 176 and 178.

and

$$
\text{(20a)} \qquad \begin{aligned} w_0 &= a_2 - i\infty, & 0 < a_2 < \pi, \\ w_1 &= b_2 + i\infty, & \pi < b_2 < 2\pi. \end{aligned}
$$

These paths, which are largely arbitrary and are restricted only asymptotically to the shaded regions, are denoted by W_1 and W_2 in Fig. 19. The fact that they have been drawn through the points $w = 0$ and $w = \pi$ is also arbitrary but will prove convenient later. The constant c_n is now determined by

$$
\text{(21)} \qquad c_n = \frac{1}{\pi} e^{-in\pi/2}.
$$

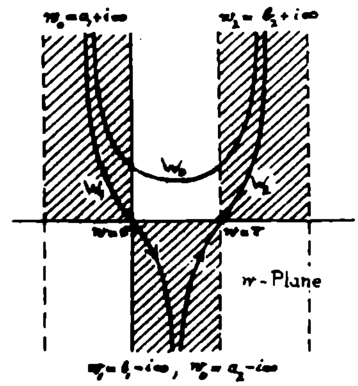


Fig. 19. The paths of integration W_1 and W_2 for H^1 and H^2. Combined in succession they are equivalent to the path W_0.

The cylindrical functions thus obtained are called the *first and second Hankel functions.*

$$
\text{(22)} \qquad \begin{aligned} H_n^1(\varrho) &= \frac{1}{\pi} \int_{W_1} e^{i\varrho\cos w}\, e^{in(w-\pi/2)}\, dw, \\ H_n^2(\varrho) &= \frac{1}{\pi} \int_{W_2} e^{i\varrho\cos w}\, e^{in(w-\pi/2)}\, dw. \end{aligned}
$$

They are almost more important to mathematical physicists than the Bessel functions I_n. They differ from the latter by the fact that they become infinite at $\varrho = 0$ even for positive n. This follows from the fact that the integral

$$
\int_{W_1, W_2} e^{in(w-\pi/2)}\, dw
$$

obtained from (22) by setting $\varrho = 0$, diverges in the infinite part of the lower half-plane.

The singularities of H^1 and H^2 at $\varrho = 0$ will be discussed in Section C. Due to their construction H^1 and H^2 are again solutions of the differential equation (11). The general integral of (11) can therefore be written in the form

$$
\text{(23)} \qquad Z_n(\varrho) = C_1 H_n^1(\varrho) + C_2 H_n^2(\varrho)
$$

We now want to show that the special integral I_n is obtained from this formula by setting

$$
C_1 = C_2 = \frac{1}{2}
$$

as is seen by looking at Fig. 19. If we traverse the paths W_1 and W_2 in succession then the lower parts cancel and the whole path contracts to W_0. Considering the new determination of c_n in (21) we have twice the amount obtained with the previous definition (13). Hence we indeed have

$$I_n(\varrho) = \frac{1}{2}\{H_n^1(\varrho) + H_n^2(\varrho)\}. \tag{24}$$

With this we compare the difference $H_n^1 - H_n^2$, which, written in terms of the variables of integration β and γ of (15), is purely imaginary for real ϱ and n. We denote this difference by $2iN_n$ and call $N_n(\varrho)$ Neumann's function:

$$N_n(\varrho) = \frac{1}{2i}\{H_n^1(\varrho) - H_n^2(\varrho)\}. \tag{25}$$

From (24) and (25) we have

$$\begin{aligned} H_n^1(\varrho) &= I_n(\varrho) + iN_n(\varrho), \\ H_n^2(\varrho) &= I_n(\varrho) - iN_n(\varrho). \end{aligned} \tag{26}$$

This decomposition of H^1, H^2 is completely analogous to the decomposition of the exponential function into its trigonometric components, as indicated in the following arrangement:

e^{ix}	e^{-ix}	$\cos x$	$\sin x$
$H^1(\varrho)$	$H^2(\varrho)$	$I(\varrho)$	$N(\varrho)$.

We shall see in Section D that this is not only a qualitative analogy, but that asymptotically (for $\varrho \to \infty$) it holds quantitatively also. Just as we prefer the exponential imaginary representation to the trigonometric real one in descriptions of wave phenomena, so as a rule we prefer a representation in terms of Hankel functions to one in terms of Bessel and Neumann functions, especially since our complex integrals are equally convenient for all three.

For non-integral n the H^1, H^2 must be expressible in the form (19). In order to determine the coefficients c_1, c_2 we make the following observation: according to (14)

$$2\pi e^{in\pi/2} I_n(\varrho) = \int_{W_0} e^{i\varrho\cos w + inw}\, dw \tag{27}$$

and if we replace n by $-n$ and w by $-w$ (or W_0 by $-W_0$),

(28) $$-2\pi e^{-in\pi/2} I_{-n}(\varrho) = \int_{-W_0} e^{i\varrho\cos w + inw}\, dw.$$

In Fig. 20 we have W_0 and $-W_0$, drawn for convenience in rectangular shape, with their proper orientation. Their central parts from $w = -\pi/2$ to $w = +\pi/2$ cancel. There remain two rectangular paths, which, for convenience, we have deformed into paths of the type W_2 in Fig. 19. The right hand path from $\frac{\pi}{2} - i\infty$ to $\frac{3\pi}{2} + i\infty$ coincides with W_2 Let the left hand path from $-\frac{\pi}{2} + i\infty$ to $-\frac{3\pi}{2} - i\infty$ be denoted by W_2'. Adding (27) and (28) we obtain then

(29) $$2\pi\{e^{in\pi/2} I_n(\varrho) - e^{-in\pi/2} I_{-n}(\varrho)\} = \left(\int_{W_2} + \int_{W_2'}\right) e^{i\varrho\cos w + inw}\, dw.$$

Here according to (22) the integral over W_2 equals

(29a) $$\pi e^{in\pi/2} H_n^2(\varrho),$$

The integral over W_2' differs from this only in the orientation of the path and in its translation by -2π. This integral, according to (22), is then

(29b) $$-\pi e^{-3in\pi/2} H_n^2(\varrho).$$

Substituting (29a,b) in (29) we obtain

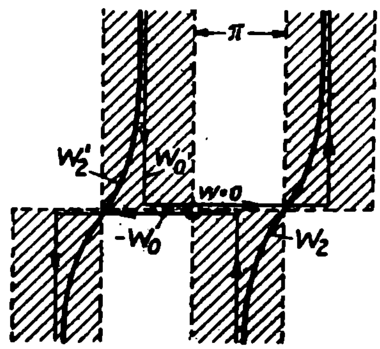


Fig. 20. The paths W_0 and $-W_0$ for I_n and I_{-n} are equivalent to the paths W_2 and W'_2 which belong to the type H^2_n.

$$2[I_n(\varrho) - e^{-in\pi} I_{-n}(\varrho)] = (1 - e^{-2in\pi}) H_n^2(\varrho)$$

and hence

(30) $$H_n^2(\varrho) = \frac{e^{in\pi} I_n(\varrho) - I_{-n}(\varrho)}{i \sin n\pi}.$$

The corresponding representation for H^1 is obtained from (24):

(31) $$H_n^1(\varrho) = 2\, I_n(\varrho) - H_n^2(\varrho) = \frac{e^{-in\pi} I_n(\varrho) - I_{-n}(\varrho)}{-i \sin n\pi}.$$

The coefficients c_1, c_2 for Hankel functions in equation (19) are thereby determined. We note that for real n and complex ϱ

(32) $$H_n^1(\varrho^*) = [H_n^2(\varrho)]^*, \qquad \text{hence} \qquad H_n^2(\varrho^*) = [H_n^1(\varrho)]^*.$$

Here the asterisk * stands as usual for the passage to the complex conjugate. In the derivation of (32) from (30) and (31) we use the relation $I_n(\varrho^*) = [I_n(\varrho)]^*$ which follows from (34). We further deduce from (30) and (31) that

$$H^1_{-n}(\varrho) = e^{in\pi} H^1_n(\varrho), \quad H^2_{-n}(\varrho) = e^{-in\pi} H^2_n(\varrho). \tag{32a}$$

and from (25), (30) and (31) that

$$N_n(\varrho) = \frac{\cos n\pi \, I_n(\varrho) - I_{-n}(\varrho)}{\sin n\pi}. \tag{33}$$

C. Series Expansion at the Origin

We have seen that $I_n(\varrho)$ is regular in the entire finite plane. It can therefore be expanded in ascending powers of ϱ. Indeed we see directly that the differential equation (11) is satisfied by the series:

$$I_n(\varrho) = \left(\frac{\varrho}{2}\right)^n \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(n+m+1)} \left(\frac{\varrho}{2}\right)^{2m}. \tag{34}$$

For $n = 0$ it assumes the particularly elegant form

$$I_0(\varrho) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,m!} \left(\frac{\varrho}{2}\right)^{2m}. \tag{35}$$

which was known to Fourier. We shall demonstrate in exercise IV.1 that these series agree with the integral representation (14).

In order to obtain the series for the general cylinder function Z_n and to investigate the *singularity* at $\varrho = 0$, we proceed as in the case of ordinary linear differential equations: we write

$$Z_n = \varrho^\lambda (a_0 + a_1 \varrho + a_2 \varrho^2 + \cdots + a_k \varrho^k + \cdots) \tag{36}$$

and substitute this in the differential equation (11); the resulting power series must vanish term by term. The lowest power $\varrho^{\lambda-2}$ yields the determination of λ, the general term $\varrho^{\lambda+k-2}$ yields a recursion formula for a_k. We obtain

$$\lambda(\lambda - 1) + \lambda - n^2 = 0, \qquad \lambda = \pm n, \tag{37}$$

and

$$\{(\lambda + k)(\lambda + k - 1) + \lambda + k - n^2\} a_k + a_{k-2} = 0, \tag{37a}$$

By the use of (37) equation (37a) can be simplified to

(37 b) $$(k^2 + 2k\lambda)a_k + a_{k-2} = 0.$$

By repeated application of this recursion formula we get for $k = 2m$

$$a_{2m} = \frac{-a_{2m-2}}{4m(m+\lambda)} = \frac{(-1)^2}{2^4} \frac{a_{2m-4}}{m(m-1)\,(m+\lambda)\,(m+\lambda-1)} = \cdots$$

If, as in (34), we choose $a_0 = 1/2^n\,\Gamma(\lambda+1)$ and set $a_1 = 0$ then we obtain

(38) $$a_{2m} = \frac{(-1)^m}{2^{2m+n}} \frac{1}{m!\,\Gamma(m+\lambda+1)}, \quad a_{2m+1} = 0\,.$$

This establishes the validity of (34). According to (37) it is equally valid for $\lambda = -n$ and for $\lambda = +n$. As mentioned on p. 88, equation (11) has the solution $I_{-n}(\varrho)$ in addition to $I_n(\varrho)$. If n has a positive real part the latter vanishes for $\varrho = 0$ with the same rapidity as ϱ^n, whereas the former becomes *infinite* with the same rapidity as ϱ^{-n}.

What we have said so far in Section C holds only when n is *non-integral.* For *integral* n, or more generally for the cases in which the difference of the two roots of (37) is integral,[4] we encounter in the solution belonging to the smaller λ a difficulty that is well known from the general theory of linear differential equations, namely, that in addition to powers with negative exponents we have *logarithmic terms.* We demonstrate this as follows.

Substituting in (37b) $\lambda = -n$ and $k = 2n$, we obtain $a_{2n-2} = 0$. Hence, tracing the recursion formula for a_{2n} backward, we see that in the series (36) for $Z_n = I_{-n}$ all the terms $a_k = a_{2n}$ vanish. This implies the relation (19a) previously established between I_n and I_{-n}.

The problem is to find a second solution of Bessel's differential equation (11) which is linearly independent of I_n. We do this by a limit consideration in which n is taken as a positive number which is arbitrarily close to an integer. Instead of applying this to the Hankel function H we apply it directly to the Neumann function N of equation (33), the decisive function for the singularity under discussion. Before passing to the limit N is given by (33); in the limit it becomes 0/0 due to equation (19a). The limiting value is determined according to De l'Hospital's rule. Denoting the integral limit of n by $\bar{n}$, we get for the denominator of (33)

$$\frac{\partial}{\partial n}\sin n\pi = \pi\cos n\pi = \pi\,(-1)^{\bar{n}}$$

[4] This is the case for Bessel functions in which n is half an integer. The fact that, in spite of this, the complications which are discussed in the text do not arise, will be explained in §21 C.

and for the numerator

$$- \pi \sin n\pi \, I_n(\varrho) + \cos n\pi \frac{\partial}{\partial n} I_n(\varrho) - \frac{\partial}{\partial n} I_{-n}(\varrho)$$
$$= (-1)^n \left\{ \frac{\partial}{\partial n} I_n(\varrho) - (-1)^n \frac{\partial}{\partial n} I_{-n}(\varrho) \right\}_{n \to \bar{n}},$$

hence

$$\pi N_{\bar{n}}(\varrho) = \lim_{n \to \bar{n}} \left\{ \frac{\partial}{\partial n} I_n(\varrho) - (-1)^n \frac{\partial}{\partial n} I_{-n}(\varrho) \right\}. \tag{39}$$

Here the limit sign indicates that the differentiation with respect to n must be performed before the passage to the limit, i.e., for non-integral n. Since we are primarily interested in the neighborhood of $\varrho = 0$, we naturally use the series (34) which (for non-integral n) represents not only I_n but also I_{-n}. We compute the two parts of the right side of (39) separately.

Using the well known formula

$$\frac{d}{dx} a^x = a^x \log a$$

we obtain from the first term of the series (34)

$$\lim \frac{\partial}{\partial n} I_n = \frac{1}{\Gamma(\bar{n}+1)} \left(\frac{\varrho}{2}\right)^{\bar{n}} \left\{ \log \frac{\varrho}{2} - \frac{\Gamma'(\bar{n}+1)}{\Gamma(\bar{n}+1)} \right\} + \cdots, \tag{40}$$

where the three dots indicate terms of higher order than $\varrho^{\bar{n}}$. We use the abbreviation introduced by Gauss

$$\Psi(z) = \frac{\Gamma'(z+1)}{\Gamma(z+1)} = -C + \sum_{\nu=1}^{\infty} \left(\frac{1}{\nu} - \frac{1}{z+\nu} \right) \tag{41}$$

where C is Euler's constant

$$C = \lim_{n \to \infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log n \right) = 0.5772.$$

If we set

$$C = \log \gamma, \quad \gamma = 1.781, \tag{41a}$$

then using (41) and (41a) we can write for the term { } in (40)

$$\log \frac{\varrho}{2} - \Psi(\bar{n}) = \log \frac{\gamma \varrho}{2} - \sum_{\nu=1}^{\bar{n}} \frac{1}{\nu}. \tag{41b}$$

The coefficient of the term { } in (40) is equal to $I_n(\varrho)$ except for higher powers than $\varrho^{\bar{n}}$. Using (41b) we can rewrite (40) as

$$\text{(42)} \qquad \operatorname{Lim} \frac{\partial}{\partial n} I_n = \left\{\log \frac{\gamma \varrho}{2} - \sum_{\nu=1}^{\bar{n}} \frac{1}{\nu}\right\} I_{\bar{n}}(\varrho) \ \ldots$$

The three dots indicate that equation (42) is exact only up to terms of order $\varrho^{\bar{n}}$.

The computation of the second term on the right side of (39) is somewhat different. We start from

$$\text{(43)} \qquad \begin{aligned} I_{-n} = \left(\frac{\varrho}{2}\right)^{-n} &\left\{\frac{1}{\Gamma(-n+1)} - \frac{1}{1!\,\Gamma(-n+2)}\left(\frac{\varrho}{2}\right)^2 + \cdots \right. \\ &\left. + \frac{(-1)^{\bar{n}}}{\bar{n}!\,\Gamma(-n+\bar{n}+1)}\left(\frac{\varrho}{2}\right)^{2\bar{n}} + \cdots\right\}. \end{aligned}$$

By first differentiating only the term $(\varrho/2)^{-n}$ with respect to n we get as in (40)

$$-\log\frac{\varrho}{2}\left(\frac{\varrho}{2}\right)^{-n}\left\{\frac{1}{\Gamma(-n+1)} - \cdots + \frac{(-1)^{\bar{n}}}{\bar{n}!\,\Gamma(-n+\bar{n}+1)}\left(\frac{\varrho}{2}\right)^{2\bar{n}}\right\} + \cdots$$

For $n \to \bar{n}$, all the Γ's become infinite except the last. We have then:

$$\text{(44)} \qquad -\log\frac{\varrho}{2}\frac{(-1)^{\bar{n}}}{\bar{n}!}\left(\frac{\varrho}{2}\right)^{\bar{n}} = (-1)^{\bar{n}+1}\log\frac{\varrho}{2}\, I_{\bar{n}}(\varrho) + \cdots$$

On the other hand the differentiation of the term { } in (43) yields[5]

$$\text{(44a)} \quad \left(\frac{\varrho}{2}\right)^{-n}\left\{\frac{\Psi(-n)}{\Gamma(-n+1)} - \frac{\Psi(-n+1)}{1!\,\Gamma(-n+2)}\left(\frac{\varrho}{2}\right)^2 + \cdots + \frac{(-1)^{\bar{n}}\Psi(-n+\bar{n})}{\bar{n}!\,\Gamma(-n+\bar{n}+1)}\left(\frac{\varrho}{2}\right)^{2\bar{n}}\right\} + \cdots$$

The function $\Psi(z)$ has simple poles at the points $z = -1, -2, -3, \ldots$ just like $\Gamma(z+1)$. According to (41) we have in the neighborhood of the ν-th pole

$$\text{(45)} \qquad \Psi(z) = -\frac{1}{z+\nu}.$$

The development of $\Gamma(z+1)$ at the same point is[6]

$$\text{(45a)} \qquad \Gamma(z+1) = \frac{(-1)^{\nu-1}}{(\nu-1)!}\frac{1}{z+\nu}.$$

[5] In (44a) two minus signs have cancelled each other. Namely for $z = -n+1$, $-n+2, \ldots$ we have

$$\frac{d}{dn}\frac{1}{\Gamma(z)} = -\frac{\Gamma'(z)}{[\Gamma(z)]^2}\frac{dz}{dn} = (-1)^2\frac{\Gamma'(z)}{[\Gamma(z)]^2} = \frac{\Psi(z-1)}{\Gamma(z)}.$$

[6] For this and the previous formulas see Jahnke-Emde's tables of functions, 3rd ed., Teubner, Leipzig, 1938, p. 10, 11, and 18.

Hence

$$\frac{\Psi(z)}{\Gamma(z+1)} = (-1)^{\nu} (\nu-1)! \quad \text{for} \quad z = -\nu. \tag{45b}$$

Since $\Gamma(1) = 1$ and $\Psi(0) = -C$ we have, in the neighborhood of $z = 0$

$$\frac{\Psi(z)}{\Gamma(z+1)} = -C = -\log\gamma. \tag{45c}$$

After these preparations we can pass to the limit in (44a). According to (45) and (45a) all the terms Ψ/Γ, with the exception of the last, have the form ∞/∞ which according to (45b) can be replaced by $(-1)^{\nu} (\nu-1)!$ where $\nu = \bar{n}$ in the first term, $\nu = \bar{n}-1$ in the subsequent terms. For the last term we apply (45c) and obtain

$$-\frac{(-1)^{\bar{n}}}{\bar{n}!}\log\gamma\left(\frac{\varrho}{2}\right)^{2\bar{n}} = (-1)^{\bar{n}}\left(\frac{\varrho}{2}\right)^{\bar{n}}\log\gamma\, I_{\bar{n}}(\varrho) + \cdots$$

We have as the limiting value of (44a) (instead of $\bar{n}$ we now write n, which is still an integer)

$$(-1)^n\left\{(n-1)!\left(\frac{\varrho}{2}\right)^{-n} + \frac{(n-2)!}{1!}\left(\frac{\varrho}{2}\right)^{-n+2} + \cdots - \log\gamma\, I_n\right\} + \cdots. \tag{46}$$

The sum of (46) and (44) now yields the second term in $\{\}$ of (39)

$$-(-1)^n \operatorname{Lim}\frac{\partial}{\partial n} I_{-n} = -(n-1)!\left(\frac{\varrho}{2}\right)^{-n} - \frac{(n-2)!}{1!}\left(\frac{\varrho}{2}\right)^{-n+2} - \cdots + \log\frac{\gamma\varrho}{2} I_n + \cdots.$$

Combining this with (42) we obtain in (39) for $n > 0$

$$\begin{aligned}\pi N_n(\varrho) = &-(n-1)!\left(\frac{\varrho}{2}\right)^{-n} - \frac{(n-2)!}{1!}\left(\frac{\varrho}{2}\right)^{-n+2} \\ &-\cdots + 2\log\frac{\gamma\varrho}{2} I_n - \sum_{\nu=1}^{n}\frac{1}{\nu} + \cdots.\end{aligned} \tag{47}$$

The terms on the right are written in decreasing order, the term with $(\varrho/2)^{-n}$ having highest order and the logarithmic term having lowest order. This implies a simple logarithmic singularity for $n = 0$; we have then:

$$\frac{\pi}{2} N_0(\varrho) = \log\frac{\gamma\varrho}{2} I_0 + \cdots, \tag{48}$$

or the complete form, which we state without proof

$$\frac{\pi}{2} N_0(\varrho) = \log\frac{\gamma\varrho}{2} I_0(\varrho) + 2\left(I_2(\varrho) - \frac{1}{2} I_4(\varrho) + \frac{1}{3} I_6(\varrho) - \cdots\right). \tag{48a}$$

According to (26) this logarithmic singularity arises in H just as it does in N. We see from this that the H_n have branch points at the origin of the complex ϱ-plane even for integral n. From (26) and (47) we see that upon continuation around the origin H_n increases by $\mp 4I_n(\varrho)$ (for details see exercise IV.2). In exercise IV.3 we shall deduce the existence of the logarithmic singularity of H_0 in a more direct, though mathematically less satisfactory, way.

D. Recursion Formulas

The $Z_n(\varrho)$ satisfy a *differential equation* in ϱ and a *difference equation* in n, for arbitrary, not necessarily integral, n. We can deduce this from our integral representation for the H and hence for arbitrary linear combinations of the H, in particular for the I and N.

Remembering that the paths of integration W_1 and W_2 in (22) are independent of n we form:

(49) $$\frac{\pi}{2}(H_{n+1} + H_{n-1}) = \int e^{i\varrho\cos w}\, e^{in(w-\pi/2)}\{\ \}_1\, dw\,,$$

(50) $$\frac{\pi}{2}(H_{n+1} - H_{n-1}) = \int e^{i\varrho\cos w}\, e^{in(w-\pi/2)}\{\ \}_2\, dw\,.$$

where

$$\{\ \}_1 = \frac{1}{2}(e^{i(w-\pi/2)} + e^{-i(w-\pi/2)}) = \sin w\,,$$

$$\{\ \}_2 = \frac{1}{2}(e^{i(w-\pi/2)} - e^{-i(w-\pi/2)}) = -i\cos w\,.$$

We may therefore write for the integrals on the right of (49), (50)

(49a) $$-\frac{1}{i\varrho}\int \frac{\partial}{\partial w}(e^{i\varrho\cos w})\cdot e^{in(w-\pi/2)}\, dw\,,$$

(50a) $$-\frac{\partial}{\partial\varrho}\int e^{i\varrho\cos w}\cdot e^{in(w-\pi/2)}\, dw\,;$$

and by integration by parts (49a) becomes

(49b) $$\frac{n}{\varrho}\int e^{i\varrho\cos w}\cdot e^{in(w-\pi/2)}\, dw\,.$$

We now can express the right sides of (49) and (50) in terms of Hankel functions of index n. These formulas are valid for both H^1 and H^2 depending on the path of integration; we may write them directly for the general cylinder function Z, which is a linear combination of the two. We have

(51) $$Z_{n+1} + Z_{n-1} = \frac{2n}{\varrho} Z_n\,,$$

and

$$Z_{n+1} - Z_{n-1} = -2\frac{dZ_n}{d\varrho}. \tag{52}$$

These are the recursion formulas we were seeking. They hold for n integral or non-integral, positive or negative.

For $n = 0$ we get as a special case

$$Z_{-1} = -Z_{+1} \tag{51a}$$

and

$$Z_1 = -\frac{dZ_0}{d\varrho}, \tag{52a}$$

and by further specialization of (52a) we get the relation

$$I_1(\varrho) = -\frac{d}{d\varrho}I_0(\varrho). \tag{52b}$$

which could also have been obtained directly from the series (27) and (27a).

E. Asymptotic Representation of the Hankel Functions

The integrand in our representations (14) and (22) oscillates more and more rapidly with increasing ϱ, for the non-shaded regions of the w-plane with increasing amplitude, for the shaded region with amplitude decreasing to zero. As shown in Fig. 19, the paths W_1 and W_2 for H^1 and H^2 can be drawn completely in the shaded regions for real ϱ. The figures illustrating exercise IV.2 show that this is no longer the case for complex ϱ. We also see from Fig. 19 that the points $w = 0$ and $w = \pi$, at which the paths touch two non-shaded regions, will play an important role for the asymptotic computation of H^1 and H^2.

We shall develop here the *method of saddle points* in an intuitive, so to speak topographical, manner, and leave all analytic refinements and generalizations for §21. We assume

$$\varrho \text{ real} \gg 1 \qquad \text{and} \qquad n \ll \varrho. \tag{53}$$

For H^1 the path W_1 begins and ends in the shaded "low lands," and the same holds for H^2 and W_2. The deciding exponent has its extremum at

$$\sin w = 0, \qquad w = \begin{cases} 0 & \text{on } W_1 \\ \pi & \text{on } W_2. \end{cases}$$

This extremum, like all extrema of real or imaginary parts of complex functions, is not a maximum or a minimum but a saddle point. To the right and left of W_1 and W_2 at these points there tower steeply rising mountain ranges. Between them run W_1 and W_2 as *mountain passes.* The saddle point method is therefore also called the *pass method.* The altitude of the paths at $w = 0$ and $w = \pi$ is

$$|e^{i\varrho}| = 1 \qquad \text{and} \qquad |e^{-i\varrho}| = 1\,.$$

What path should a mountain climber take in order to surmount the pass in the fastest possible manner? The answer is, the path of steepest ascent and descent, the so-called "drop lines." However this prescription is not binding and it may be amended for reasons of convenience (analytic reasons[7] or mountain climber's reasons). The English name "method of steepest descent" instead of pass method is therefore not entirely appropriate.

We consider a short segment of the path W_1 in the neighborhood of the crest of the path: let ds denote the arc element on this path with the orientation W_1, and let the crest itself be given by $s = 0$. We write:

$$(54) \qquad w = s\,e^{i\gamma}, \quad i\cos w = i\Big(1 - \frac{s^2}{2}e^{2i\gamma}\Big) = \frac{s^2}{2}\sin 2\gamma + i\Big(1 - \frac{s^2}{2}\cos 2\gamma\Big).$$

The level lines of the real part are perpendicular to both the level lines of the imaginary part and to the drop lines, therefore the level line of the imaginary part is at the same time the drop line of the real part which determines the altitude of the pass. In our case the level line of the imaginary part of (54) is given by

$$1 - \frac{s^2}{2}\cos 2\gamma = \text{const.}$$

with the constant equal to one, since the line must pass through the crest $s = 0$. Hence we have

$$(54\text{a}) \qquad \cos 2\gamma = 0, \qquad \gamma = \mp\frac{\pi}{4}.$$

For H^1 we must choose the minus sign for γ (see Fig. 19) whereby (54) becomes

$$(54\text{b}) \qquad w = e^{-i\pi/4}s, \qquad dw = e^{-i\pi/4}\,ds, \qquad i\cos w = i - \frac{s^2}{2}.$$

We substitute this in (22) and at the same time set $s = 0$ in the "slowly varying" factor $\exp\{in(w - \pi/2)\}$; the integration can obviously

[7] G. Faber, Bayr. Akad. 1922, p. 285.

be restricted to the immediate neighborhood of the pass, say to distances $< \varepsilon$. We obtain

$$(54c) \qquad H_n^1(\varrho) = \frac{1}{\pi} e^{i[\varrho - (n + \frac{1}{2})\pi/2]} \int_{-\varepsilon}^{+\varepsilon} e^{-\frac{\varrho}{2}s^2} ds.$$

This integral can be reduced to the Laplace integral with the help of the substitution $s = \sqrt{\frac{2}{\varrho}}\, t$, which, at the limits of integration, becomes $\sqrt{\frac{\varrho}{2}}\varepsilon \to \infty$ and $-\sqrt{\frac{\varrho}{2}}\varepsilon \to -\infty$. We therefore have the final result:

$$(55) \qquad H_n^1(\varrho) = \sqrt{\frac{2}{\varrho\pi}}\, e^{i[\varrho - (n + \frac{1}{2})\pi/2]}.$$

For H^2, where we have to use the path W_2 with the saddle point at $w = \pi$ and where in (54a) we have to choose the plus sign for γ, we obtain correspondingly

$$(56) \qquad H_n^2(\varrho) = \sqrt{\frac{2}{\varrho\pi}}\, e^{-i[\varrho - (n + \frac{1}{2})\pi/2]}.$$

By taking half the sum of (55) and (56) we get

$$(57) \qquad I_n(\varrho) = \sqrt{\frac{2}{\varrho\pi}} \cos\left[\varrho - \left(n + \frac{1}{2}\right)\frac{\pi}{2}\right].$$

These asymptotic representations, though derived for real ϱ, can be continued analytically in the complex ϱ-plane (for the representation of the two H's this plane must be cut along a suitable half line because of the branching discussed at the end of Section C). On the basis of equations (55) and (56) we state: *H^1 vanishes asymptotically for* $\operatorname{Im} \varrho \to +\infty$, *$H^2$ for* $\operatorname{Im} \varrho \to -\infty$. This is the reason for the particular suitability of Hankel functions for the treatment of problems of damped oscillations. On the other hand both the Bessel function I and the Neumann function N become asymptotically infinite in both half planes.

We shall show in §21 how our asymptotic limits can be extended into asymptotic expansions and how the condition $n \ll \varrho$ of (53) can be dropped. The factor $\varrho^{-\frac{1}{2}}$ in (55) and (56) is connected with the fact that $H^1{}_0$ (or, for another choice of time dependence, $H^2{}_0$) represents, upon introduction of a coordinate z which is perpendicular to the r, φ-plane, an *expanding cylindrical wave* with source $r = 0$. Since the energy $2\pi r\,|H_0|^2$ passing through a cylinder of radius r must be independent of r (in the absence of absorption), we see that H_0 is proportional to $r^{-\frac{1}{2}}$ or in other words to $\varrho^{-\frac{1}{2}}$.

One may think of the real and imaginary parts of $H^1{}_0$ and I_0 as defining surfaces over the complex ϱ-plane. The surface of Re $\{H_0^1\}$ osculates the positive ϱ-half-plane exponentially and in the negative half plane it has exponentially rising mountain ranges separated by correspondingly deepening valleys. The surface of Im (H_0^1) behaves similarly and in addition has a narrow funnel at the origin which corresponds to the logarithmic singularity of H_0 (also of N_0: see (48)), as well as a discontinuity along the negative real axis corresponding to the branching discussed above. The The surface of $\mathrm{Re}(I_0)$consists of a mildly undulating depression flanked on both sides by rising mountain country. The undulating nature of the depression follows from the asymptotic equation (57) and indicates the existence of an infinity of roots of the equation $I_0 = 0$ along the real axis. These roots are represented in Fig. 21. The surface of Im (I_0) is very similar in appearance except that the bottom of the depression is level throughout, corresponding to the fact that I_0 is real along the real axis.

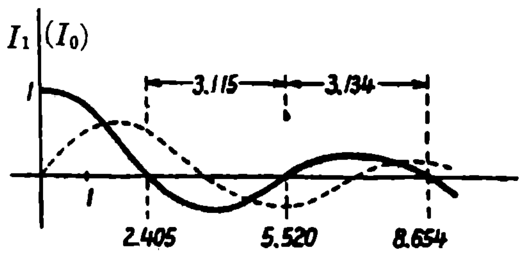


Fig. 21. Representation of I_0 (heavy line) and of I_1 (dotted line) along the real axis. The first three roots of $I_0(\varrho) = 0$.

§ 20. Heat Equalization in a Cylinder

As an excellent example for the application of the theory of Bessel functions we again consider a special problem of heat conduction. The problem was treated by Fourier, who, in fact, mentioned the functions with integral n whence they are sometimes referred to as *Fourier-Bessel functions.*

We shall treat our problem in three steps:

A. For an infinitely long cylinder and an axially symmetric initial state $f = f(r)$.

B. For an initial state which depends also on the argument $f = f(r, \varphi)$.

C. For a cylinder of finite length and general initial state $f = f(r, \varphi, z)$.

The boundary condition shall, for the sake of simplicity, always be that of isothermy

$$(1) \qquad u = 0 \qquad \text{for } r = a = \text{radius of cylinder.}$$

For the complete cylinder this is augmented by the further "boundary condition" of finality along the axis:

$$(1a) \qquad u \neq \infty \qquad \text{for} \quad r = 0.$$

A. One-Dimensional Case $f = f(r)$

The equation of heat conduction is:

$$(2) \qquad \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} = \frac{1}{k}\frac{\partial u}{\partial t}.$$

Making the special substitution

$$(3) \qquad u = R(r)\, e^{-\lambda^2 k t}$$

we get the differential equation for R

$$(3a) \qquad \frac{d^2 R}{dr^2} + \frac{1}{r}\frac{dR}{dr} + \lambda^2 R = 0.$$

This is Bessel's differential equation (19.11) with $n = 0$ and $\varrho = \lambda r$. Its general solution can be written as:

$$Z_0 = A\, I_0(\lambda r) + B N_0(\lambda r).$$

However, the condition of finality (1a) requires that we set $B = 0$; because of (1) we must further demand that

$$(4) \qquad I_0(\lambda a) = 0.$$

We already know that this equation has an infinity of roots such that the distance between consecutive roots approaches π; from (19.57) we get for the m-th root

$$(4a) \qquad \lambda_m a - \frac{\pi}{4} = \left(m - \frac{1}{2}\right)\pi, \quad \text{hence} \quad \lambda_m a = \left(m - \frac{1}{4}\right)\pi.$$

This approximation is valid down to $m = 2$ with an accuracy of about 1%; for $m = 1$ we get

$$(4b) \qquad \lambda_1 a = 2.40$$

as compared to 2.36 from (4a) (see Fig. 21).

We have then at our disposal an infinity of solutions of (3a):

$$R(r) = A_m I_0(\lambda_m r), \qquad m = 1, 2, \ldots .$$

Correspondingly we get from (3) as the general solution of our problem

(5) $$u = \sum_{m=1}^{\infty} A_m I_0 (\lambda_m r) e^{-\lambda_m^2 k t}.$$

We now merely need to satisfy the initial condition

(6) $$f(r) = \sum_{m=1}^{\infty} A_m I_0 (\lambda_m r).$$

A way of doing this is indicated by the treatment of the anharmonic sine series in §16. In order to emphasize the complete analogy with the equations (5) and (6) of §16 we write

$$u_n = I_0 (\lambda_n r), \qquad u_m = I_0 (\lambda_m r)$$

and then write our present equation (3a) in the form:

$$\frac{d}{dr}\left(r \frac{du_n}{dr}\right) + \lambda_n^2 r u_n = 0, \qquad \frac{d}{dr}\left(r \frac{du_m}{dr}\right) + \lambda_m^2 r u_m = 0.$$

Multiplying by u_m and u_n and subtracting we get as an analogue to (16.5a)

(7) $$u_m \frac{d}{dr}\left(r \frac{du_n}{dr}\right) - u_n \frac{d}{dr}\left(r \frac{du_m}{dr}\right) = (\lambda_m^2 - \lambda_n^2) r u_m u_n.$$

Integrating over the fundamental domain $0 < r < a$ we get as an analogue to (16.6)

(7a) $$(\lambda_m^2 - \lambda_n^2) \int_0^a r u_m u_n dr = r \left(u_m \frac{du_n}{dr} - u_n \frac{du_m}{dr}\right) \Big|_0^a.$$

This is *Green's theorem* applied to the two-dimensional circular region $r = a$.

The right side of (7a) vanishes for the upper limit $r = a$ on account of equation (1), for the lower limit $r = 0$ on account of the factor r and equation (1a). Since $\lambda_m \neq \lambda_n$, for $m \neq n$, we have the *orthogonality condition:*

(8) $$\int_0^a u_m u_n r \, dr = 0 \text{ for } m \neq n.$$

The "weighting factor" r is due to the two-dimensional element of area $r \, dr \, d\varphi$ in Green's theorem.

From (7a) we may also deduce the normalizing integral

$$N_m = \int u_m^2 r \, dr$$

if we drop the assumption that λ_n is a root of (4). We consider λ_n

rather as a continuous variable which in the limit coincides with λ_m. Equation (7a) then represents N_m as a fraction which for $\lambda_n \to \lambda_m$ assumes the form 0/0. By differentiating the numerator and denominator with respect to λ_n and substituting $r = a$ and $r = 0$ we find, because of (1), that

$$(9) \qquad N_m = \frac{a}{2\lambda_n}\left(\frac{du_n}{d\lambda_n}\frac{du_m}{dr}\right)_{r=a} \to \frac{a}{2\lambda_m}\left(\frac{du_m}{d\lambda_m}\frac{du_m}{dr}\right)_{r=a}.$$

But for $r = a$

$$\frac{du_m}{d\lambda_m} = a\, I_0'(\lambda_m a), \qquad \frac{du_m}{dr} = \lambda_m\, I_0'(\lambda_m a).$$

Substituting this in (9) we get

$$(9a) \qquad N_m = \frac{a^2}{2}[I_0'(\lambda_m a)]^2.$$

The coefficients A_m of the series (6) can now be calculated from (8) and (9) in the Fourier manner:

$$(10) \qquad A_m N_m = \int_0^a f(r)\, I_0(\lambda_m r)\, r\, dr,$$

We substitute this in the series (5), thereby completing the solution of problem A.

B. Two-Dimensional Case $f = f(r, \varphi)$

We first develop $f(r, \varphi)$ in the complex Fourier series (1.12)

$$(11) \quad f(r,\varphi) = \sum_{n=-\infty}^{n=+\infty} C_n e^{in\varphi}, \quad C_n = C_n(r) = \frac{1}{2\pi}\int_{-\pi}^{+\pi} f(r,\varphi)\, e^{-in\varphi}\, d\varphi.$$

Due to the two-dimensional equation (2)

$$(12) \qquad \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \varphi^2} = \frac{1}{k}\frac{\partial u}{\partial t}$$

and the generalized substitution (3)

$$(13) \qquad u = R_n(r)\, e^{in\varphi}\, e^{-\lambda^2 kt}$$

we have the differential equation for $R_n(r)$

$$(14) \qquad \frac{d^2 R_n}{dr^2} + \frac{1}{r}\frac{dR_n}{dr} + \left(\lambda^2 - \frac{n^2}{r^2}\right) R_n = 0.$$

This is Bessel's differential equation (19.11) with $\varrho = \lambda r$. Equation (1a) requires that the only permissible solutions be of the form $A_n I_n(\lambda r)$. On account of (1) λ must satisfy the equation $I_n(\lambda a) = 0$ which, just like $I_0(\lambda a) = 0$, has an infinity of roots:

$$\lambda_{n,1}, \ \lambda_{n,2}, \ldots, \lambda_{n,m}, \ldots .$$

Each of these roots yields a particular solution of the form (13):

$$u_{n,m} = A_{n,m} I_n(\lambda_{n,m} r)\, e^{in\varphi}\, e^{-\lambda_{n,m} kt}, \tag{15}$$

and these solutions satisfy the differential equation (12). Through superposition we can construct from them the general solution of (14) which at the same time satisfies our boundary conditions:

$$u = \Sigma\,\Sigma\, u_{n,m} = \sum_{n=-\infty}^{+\infty} \sum_{m=1}^{\infty} A_{n,m} I_n(\lambda_{n,m} r)\, e^{in\varphi}\, e^{-\lambda^2_{n,m} kt}. \tag{16}$$

Here the constants $A_{n,m}$ must be chosen so that for $t = 0$ and every integer $-\infty < n < +\infty$ we have the equation

$$C_n(r) = \sum_{m=1}^{\infty} A_{n,m} I_n(\lambda_{n,m} r) \tag{17}$$

where according to (11) the left side is a known function of r. Equation (17) necessitates the development of this function in Bessel functions I_n. This is possible due to the *orthogonality* of the latter, which follows from Bessel's differential equation (14) and Green's theorem as in (7) and (7a)[8]. Using the abbreviations

$$v_m = I_n(\lambda_{n,m} r), \qquad v_l = I_n(\lambda_{n,l} r)$$

we obtain as generalization of (7)

$$(\lambda^2_{n,m} - \lambda^2_{n,l}) \int_0^a r\, v_m v_l\, dr = r\left(v_m \frac{dv_l}{dr} - v_l \frac{dv_m}{dr}\right)\Big|_0^a$$

Here, too, the right side vanishes. We thus have for $l \neq m$

$$\int_0^a v_m\, v_l\, r\, dr = 0. \tag{18}$$

At the same time we obtain by a passage to the limit as described in (9)

[8] In order to avoid the trivial result $0 = 0$, in the application of Green's theorem to the circle $r = a$ in the r, φ- plane we must use the two functions

$$v_{nm} = I_n(\lambda_{n,m} r)\, e^{+in\varphi} \quad \text{and} \quad v_{nl} = I_n(\lambda_{n,l} r)\, e^{-in\varphi}$$

$$N_{n,m} = \int_0^a v_m^2 r\, dr = \frac{a^2}{2} [I_n' (\lambda_{n,m} a)]^2 . \tag{19}$$

The A_{nm} in (17) can now be calculated in the Fourier manner from the given $C_n(r)$ by (18) and (19) in analogy to (10). Substituting these expressions for $C_n(r)$ in (11) we obtain

$$2\pi N_{n,m} A_{n,m} = \int_0^a \int_{-\pi}^{+\pi} f(r,\varphi)\, I_n(\lambda_{n,m} r)\, e^{-in\varphi}\, r\, dr\, d\varphi . \tag{20}$$

which concludes the solution of (16).

C. Three-Dimensional Case $f = f(r, \varphi, z)$

Let the cylinder have the finite length h and let $0 < z < h$. We first develop $f(r,\varphi,z)$ in a Fourier series with respect to z, which, due to the boundary conditions $u = 0$ for $z = 0$ and $z = h$, becomes a pure sine series:

$$f(r,\varphi,z) = \sum_{\mu=1}^{\infty} B_\mu \sin \mu\pi \frac{z}{h}, \qquad B_\mu = \frac{2}{h} \int_0^h f(r,\varphi,z) \sin \mu\pi \frac{z}{h}\, dz ; \tag{21}$$

We then develop $B_\mu = B_\mu(r,\varphi)$ in a series of $\exp(in\varphi)$:

$$B_\mu(r,\varphi) = \sum_{n=-\infty}^{+\infty} C_{\mu,n}\, e^{in\varphi}, \quad C_{\mu,n} = \frac{1}{2\pi} \int_{-\pi}^{+\pi} B_\mu(r,\varphi)\, e^{-in\varphi}\, d\varphi \tag{22}$$

Finally we represent $C_{\mu,n} = C_{\mu,n}(r)$ as a series in the Bessel functions $I_n(\lambda r)$, which progresses according to the roots of

$$I_n(\lambda a) = 0, \qquad \lambda = \lambda_{n,m}, \qquad m = 1, 2, \ldots$$

Due to the three-dimensional equation of heat conduction

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \varphi^2} + \frac{\partial^2 u}{\partial z^2} = \frac{1}{k}\frac{\partial u}{\partial t}$$

the time factor has the form

$$e^{-\alpha^2 kt} \qquad \text{with} \quad \alpha^2 = \lambda_{n,m}^2 + \left(\frac{\mu\pi}{h}\right)^2 .$$

The complete solution is given by the triply infinite sum

$$(23)\quad u=\sum_{\mu=1}^{\infty}\sum_{n=-\infty}^{\infty}\sum_{m=1}^{\infty}A_{\mu n m}\,I_n(\lambda_{n,m}\,r)\,e^{in\varphi}\sin\mu\pi\frac{z}{h}\,e^{-\left[\lambda_{n,m}^2+\left(\frac{\mu\pi}{h}\right)^2\right]kt}$$

The coefficients A are calculated from the $N_{n,m}$ of (19) in analogy to (10) and (20)

$$(24)\quad \pi h\,N_{n,m}\,A_{\mu,n,m}=\int_0^a\int_{-\pi}^{+\pi}\int_0^h f(r,\varphi,z)\,I_n(\lambda_{n,m}\,r)\,e^{-in\varphi}\sin\mu\pi\frac{z}{h}\,r\,dr\,d\varphi\,dz\,.$$

This completes the solution of (23).

In the case of a *hollow cylinder* the condition of finality (1a) is dropped. Hence in the expansion of the solution there may appear terms with N_n in addition to those with I_n (or, in other words, terms in $H^1{}_n$ and $H^2{}_n$). Heat conduction through a heating pipe is an example of this.

§ 21. More About Bessel Functions

A. Generating Function and Addition Theorems

In §19 we started from the two-dimensional wave equation $\Delta u+k^2u=0$ and its simplest solution, the *plane wave*

$$(1)\qquad u=e^{ikx}=e^{i\varrho\cos\varphi},\qquad \varrho=k\,r,\qquad k=\text{ the wave number.}$$

If we develop this into a Fourier series then, due to the origin of Bessel's differential equation (19.11), the coefficients must be Bessel functions, and because of the regularity of (1) for $r=0$ only the I functions will appear. Hence, we set the coefficient of $\exp(i\,n\,\varphi)$ equal to c_nI_n and according to (1.12) we then have

$$c_n\,I_n(\varrho)=\frac{1}{2\pi}\int_{-\pi}^{+\pi}e^{i\varrho\cos\varphi}\,e^{-in\varphi}\,d\varphi.$$

If we compare this with (19.18), in which we may replace w by $-w$, we obtain

$$c_n=e^{in\pi/2}.$$

Hence we have the Fourier series

(2)
$$e^{i\varrho\cos\varphi} = \sum_{n=-\infty}^{+\infty} e^{in\pi/2} I_n(\varrho)\, e^{in\varphi}$$

or upon the substitution $\psi = \varphi + \pi/2$

(2a)
$$e^{i\varrho\sin\psi} = \sum_{n=-\infty}^{+\infty} I_n(\varrho)\, e^{in\psi}.$$

In the older literature (2) is usually written in the less symmetric form

(2b)
$$e^{i\varrho\cos\varphi} = I_0(\varrho) + 2\sum_{n=1}^{\infty} i^n I_n(\varrho)\cos n\varphi.$$

The left sides of (2) and (2a) are called *generating functions of the Bessel functions with integral index.*

We now pass from the case of a plane wave to that of a *cylindrical wave* with its logarithmic source at the origin, which, according to p. 100, is represented by $H_0(\varrho)$. (We omit the upper index since the following is valid for both functions H, i.e., both for radiated and for absorbed waves.) We now shift the origin from $\varrho = 0$ to $\varrho = \varrho_0, \varphi = \varphi_0$ whereby $H_0(\varrho)$ goes over into

$$H_0(R), \qquad R = \sqrt{\varrho^2 + \varrho_0^2 - 2\varrho\varrho_0\cos(\varphi - \varphi_0)}.$$

If we develop this into a Fourier series with respect to $\varphi - \varphi_0$, then the coefficients must again be cylinder functions, namely functions $H_n(\varrho)$ for $\varrho < \varrho_0$ and functions $I_n(\varrho)$ for $\varrho > \varrho_0$. The latter follows from the fact that $\varrho = 0$ is now a regular point, the former from the fact that each term in the series must have the same type of radiation or absorption for $\varrho \to \infty$ as $H_0(R)$ itself. For reasons of symmetry the same consideration holds for the dependence of our coefficients on the variable ϱ_0, except that the functions I_n and H_n are interchanged since the condition $\varrho \gtrless \varrho_0$ is the same as $\varrho_0 \lessgtr \varrho$. Hence the n-th Fourier coefficient must be

$$c_n \begin{cases} I_n(\varrho_0)\, H_n(\varrho) & \text{for } \varrho > \varrho_0, \\ H_n(\varrho_0)\, I_n(\varrho) & \text{for } \varrho < \varrho_0. \end{cases}$$

The numerical factor c_n is independent of ϱ and ϱ_0 and is the same for both expansions, since the two series must go into each other continuously for $\varrho = \varrho_0$ (unless at the same time $\varphi = \varphi_0$, in which case both series diverge); it turns out to be equal to 1 if, in the case $\varrho < \varrho_0$, we pass to the limiting case of a plane wave $\varrho_0 \to \infty$, setting

$\varphi_0 = \pi$ and comparing the resulting asymptotic formula with equation (2). We thus obtain the *addition theorem:*

$$H_0(R) = \begin{cases} \sum_{n=-\infty}^{+\infty} I_n(\varrho_0)\, H_n(\varrho)\, e^{in(\varphi-\varphi_0)} & \varrho > \varrho_0, \\ \sum_{n=-\infty}^{+\infty} H_n(\varrho_0)\, I_n(\varrho)\, e^{in(\varphi-\varphi_0)} & \varrho < \varrho_0. \end{cases} \tag{3}$$

If we consider this written for *both* Hankel functions and take half the sum then we obtain the *addition theorem for Bessel functions:*

$$I_0(R) = \sum_{n=-\infty}^{+\infty} I_n(\varrho_0)\, I_n(\varrho)\, e^{in(\varphi-\varphi_0)} \qquad \varrho \gtrless \varrho_0. \tag{3a}$$

In the same manner we get from half the difference the additic theorem for Neumann functions, where we again have to distinguish between the cases $\varrho > \varrho_0$ and $\varrho < \varrho_0$.

Concerning (3) we note that the series in I_n corresponds to *Taylor's series* in the theory of complex functions, whereas the series in H_n corresponds to *Laurent's series.*[9] This is illustrated by the following example, in which one may replace z and z_0 by $\varrho e^{i\varphi}$ and $\varrho_0 e^{i\varphi_0}$:

$$\begin{aligned} \frac{z}{z-z_0} &= \sum_{n=0}^{\infty} z_0^n z^{-n} & |z| > |z_0|, \\ \frac{z_0}{z_0-z} &= \sum_{n=0}^{\infty} z_0^{-n} z^n & |z| < |z_0|. \end{aligned} \tag{4}$$

In §24 we shall develop corresponding addition theorems for spherical waves in space; there will also be a counterpart to the representation (2) of a plane wave.

B. Integral Representations in Terms of Bessel Functions

We shall give here the development of a given function $f(r)$ in terms of Bessel functions which is analogous to a representation by a Fourier integral. According to (12.11b) a function of two variables can be represented by the Fourier integral

$$f(x,y) = \frac{1}{(2\pi)^2} \iint_{-\infty}^{+\infty} d\omega\, d\omega' \iint_{-\infty}^{+\infty} d\xi\, d\eta\, f(\xi,\eta)\, e^{i\omega(x-\xi)+i\omega'(y-\eta)}. \tag{5}$$

[9] This is further discussed, together with questions of convergence, in §2 of the author's work which was cited on p. 80. We also refer to the great work of H. Weber, *Math. Ann.* **I**, p. 1, which was a fitting beginning for that journal. It is the problem of adapting the methods of Riemann's dissertation, i.e., of adapting the theory of the differential equation $\Delta u = 0$, to the differential equation $\Delta u + k^2 u = 0$. .

We introduce the polar coordinates:

$$\begin{aligned} x &= r\cos\varphi & \xi &= \varrho\cos\psi & \omega &= \sigma\cos\alpha \\ y &= r\sin\varphi & \eta &= \varrho\sin\psi & \omega' &= \sigma\sin\alpha \\ & & d\xi\, d\eta &= \varrho\, d\varrho\, d\psi & d\omega\, d\omega' &= \sigma\, d\sigma\, d\alpha. \end{aligned}$$

We assume the special angular dependence of $f(x,y)$:

(6) $$f(x, y) = f(r)\, e^{in\varphi} \quad (n \text{ integer}).$$

By using the relations

$$\omega\, x + \omega'\, y = \sigma\, r\cos(\alpha - \varphi)$$
$$\omega\, \xi + \omega'\, \eta = \sigma\, \varrho\cos(\psi - \alpha)$$

we can transform (5) into

(7) $$f(r)\, e^{in\varphi} = \int_0^\infty \sigma\, d\sigma \int_0^\infty f(\varrho)\, \varrho\, d\varrho \frac{1}{2\pi}\int_{-\pi}^{+\pi} e^{i\sigma r\cos(\alpha-\varphi)}\, d\alpha \frac{1}{2\pi}\int_{-\pi}^{+\pi} e^{in\psi} e^{-i\sigma\varrho\cos(\psi-\alpha)}\, d\psi.$$

In order to compare these integrals with respect to α and ψ to the representation (19.18)

$$I_n(z) = \frac{1}{2\pi}\int_{-\pi}^{+\pi} e^{iz\cos\beta}\, e^{in(\beta-\pi/2)}\, d\beta$$
$$= \frac{1}{2\pi}\int_{-\pi}^{+\pi} e^{-iz\cos\beta'}\, e^{in(\beta'+\pi/2)}\, d\beta' \qquad (\beta' = \beta - \pi)$$

we multiply under the integrals of (7) by

$$e^{in(\alpha-\pi/2)} \text{ and } e^{in(-\alpha+\pi/2)},$$

and divide through by $e^{in\varphi}$. We thus obtain the simple representation

(8) $$f(r) = \int_0^\infty \sigma\, d\sigma \int_0^\infty f(\varrho)\, I_n(\sigma r)\, I_n(\sigma\varrho)\, \varrho\, d\varrho.$$

In analogy to the form (4.13) of the Fourier integral theorem we can write this relation in the symmetric form:

(8a) $$\begin{aligned} f(r) &= \int_0^\infty \sigma\, d\sigma\, \varphi(\sigma)\, I_n(\sigma r), \\ \varphi(\sigma) &= \int_0^\infty \varrho\, d\varrho\, f(\varrho)\, I_n(\sigma\varrho). \end{aligned}$$

The transition from rectangular coordinates to polar coordinates in the passage from (5) to (7) requires certain conditions about the behavior of f at infinity which we do not discuss here. Equation (7) will be useful in the treatment of spherical waves in §24.

We obtain a further application of equation (8) if we let $f(r)$ degenerate to a δ-function, namely,[10] let

$$(9)\quad f(r)=\delta(r\,|\,s)=\begin{Bmatrix}0 & \text{for } r\neq s\\ \infty & \text{for } r=s\end{Bmatrix},\qquad \text{with } \int\limits_{s-\epsilon}^{s+\epsilon} f(r)\,r\,dr=1.$$

We then obtain from (8)

$$(9a)\qquad \int\limits_0^\infty I_n(\sigma r)\,I_n(\sigma s)\,\sigma\,d\sigma \doteq \delta(r\,|\,s)\,.$$

This equation represents the *orthogonality* of the two functions I_n at the points r and s of the continuous domain $0<{r \atop s}<\infty$; it is a counterpart to (20.18) in which we deal with two points m and l of the discrete λ-sequence. We shall return to the important relation (9a) in §36.

C. The Indices $n+1/2$ and $n\pm 1/3$

Substituting $n = 1/2$ in (19.34) we get $\Gamma(\frac{1}{2})=\sqrt{\pi}, \Gamma(\frac{3}{2})=\frac{1}{2}\sqrt{\pi}, \ldots$

$$(10)\quad \begin{aligned} I_{\frac12}(\varrho) &= \sqrt{\frac{\varrho}{2}}\left[\frac{2}{\sqrt{\pi}}-\frac{2\cdot 2}{1\cdot 3\sqrt{\pi}}\left(\frac{\varrho}{2}\right)^2+\frac{2\cdot 2\cdot 2}{2!\,1\cdot 3\cdot 5\cdot\sqrt{\pi}}\left(\frac{\varrho}{2}\right)^4-\cdots\right]\\ &= \sqrt{\frac{2\varrho}{\pi}}\left(1-\frac{\varrho^2}{3!}+\frac{\varrho^4}{5!}-\cdots\right)=\sqrt{\frac{2\varrho}{\pi}}\,\frac{\sin\varrho}{\varrho}\,;\end{aligned}$$

in the same manner we find for $n = -1/2$

$$(10a)\qquad I_{-\frac12}(\varrho)=\sqrt{\frac{2\varrho}{\pi}}\,\frac{\cos\varrho}{\varrho}\,.$$

We write generally

$$(11)\qquad I_{n+\frac12}(\varrho)=\sqrt{\frac{2\varrho}{\pi}}\,\psi_n(\varrho),\qquad \text{in particular}\quad \psi_0=\frac{\sin\varrho}{\varrho}.$$

From Bessel's differential equation (19.11) we get the differential equation for ψ_n

[10] We draw the reader's attention to the weighting factor r in the integral of (9). Because of this factor we no longer have $\int\delta(r|s)\,ds=1$, but instead $\int\delta(r|s)\,r\,dr=1$, as in equation (9).

(11a) $$\frac{1}{\varrho}\frac{d^2(\varrho\psi_n)}{d\varrho^2}+\left(1-\frac{n(n+1)}{\varrho^2}\right)\psi_n=0.$$

We shall meet this equation again in the theory of spherical harmonics. We now wish to show that the solutions which are finite for $\varrho = 0$ can be obtained from ψ_0 by the following rule:

(12) $$\psi_n=(-\varrho)^n\left(\frac{d}{\varrho\,d\varrho}\right)^n\psi_0.$$

We start from the series (19.34). Let p be an arbitrary index (in our case we have $p = 1/2$) then we have

(13) $$\frac{I_p(\varrho)}{(\varrho/2)^p}=\sum_{m=0}^{\infty}\frac{(-1)^m}{m!\,\Gamma(p+m+1)}\frac{(\varrho^2/2)^m}{2^m}.$$

We differentiate this equation m times with respect to $\varrho^2/2$. Then the right side becomes

$$\sum_{m=n}^{\infty}\frac{(-1)^m}{(m-n)!\,\Gamma(p+m+1)}\frac{(\varrho^2/2)^{m-n}}{2^m}.$$

By introducing a new index of summation ($u = m-n$, $m = n+u$) we get

(13a) $$\frac{(-1)^n}{2^n}\sum_{\mu=0}^{\infty}\frac{(-1)^\mu}{\mu!}\frac{(\varrho^2/4)^\mu}{\Gamma(\mu+n+p+1)}=\frac{(-1)^n}{2^n}\frac{I_{n+p}(\varrho)}{(\varrho/2)^{n+p}}.$$

At the same time the left side of (13) becomes (because of $d(\varrho^2/2) = \varrho\,d\varrho$)

(13b) $$\left(\frac{d}{\varrho\,d\varrho}\right)^n\frac{I_p(\varrho)}{(\varrho/2)^p}.$$

Comparing (13a) and (13b) we get

(13c) $$\frac{I_{n+p}(\varrho)}{\varrho^p}=(-\varrho)^n\left(\frac{d}{\varrho\,d\varrho}\right)^n\frac{I_p(\varrho)}{\varrho^p};$$

due to (10); this coincides with (12) for $p = 1/2$.

If instead of the *ratio* of $I_p(\varrho)$ and $(\varrho/2)^p$ we differentiate their *product* n times with respect to $\varrho^2/2$, then instead of (13c) we get

(13d) $$I_{-n+p}(\varrho)\cdot\varrho^p=\varrho^n\left(\frac{d}{\varrho\,d\varrho}\right)^n\{I_p(\varrho)\cdot\varrho^p\}.$$

If we again set $p = 1/2$ and apply definition (11) we get as the complement of (12)

$$\psi_{-n}(\varrho) = \varrho^{n-1}\left(\frac{d}{\varrho\, d\varrho}\right)^n \{\varrho\, \psi_0\}. \tag{13e}$$

From (12) and (13e) we deduce the *recursion formulas* for $\psi_{\pm n}$

$$\begin{aligned} \psi_{n+1} &= -\frac{d\psi_n}{d\varrho} + \frac{n}{\varrho}\, \psi_n, \\ \psi_{-n-1} &= \frac{d\psi_{-n}}{d\varrho} - \frac{n-1}{\varrho}\, \psi_{-n}. \end{aligned} \tag{14}$$

The corresponding formulas for $Z_{\pm n}$ were discussed in §19 D.

According to (14) we get successively from $\psi_0 = \sin\varrho/\varrho$:

$$\begin{aligned} &\psi_1 = \frac{\sin\varrho - \varrho\cos\varrho}{\varrho^2}, \qquad \psi_2 = \frac{3(\sin\varrho - \varrho\cos\varrho) - \varrho^2 \sin\varrho}{\varrho^3}, \ldots \\ &\psi_{-1} = \frac{\cos\varrho}{\varrho}, \qquad \psi_{-2} = -\frac{\cos\varrho + \varrho\sin\varrho}{\varrho^2}, \\ &\psi_{-3} = \frac{3(\cos\varrho + \varrho\sin\varrho) - \varrho^2\cos\varrho}{\varrho^3}, \ldots \end{aligned} \tag{14a}$$

We see from this that for integral n all $\psi_{\pm n}$ can be expressed in an *elementary form with the help of the sine and cosine functions.* This representation confirms the non-logarithmic character of the half-index Bessel functions, as stated in footnote 4 of this chapter.

The "Hankel functions" ζ_n which correspond to the ψ_n are given by the equation

$$H_{n+\frac{1}{2}}(\varrho) = \sqrt{\frac{2\varrho}{\pi}}\, \zeta_n(\varrho) \tag{15}$$

analogous to (11) (the upper indices 1 and 2 on both sides have been omitted). We are particularly interested in the functions ζ_0. We obtain them from the functions $H_{\frac{1}{2}}$, which we get from (19.31), (19.30), and (11):

$$H^1_{\frac{1}{2}} = \frac{-i\, I_{\frac{1}{2}} - I_{-\frac{1}{2}}}{-i} = \sqrt{\frac{2\varrho}{\pi}}\, \frac{e^{i\varrho}}{i\varrho}, \qquad H^2_{\frac{1}{2}} = \frac{i\, I_{\frac{1}{2}} - I_{-\frac{1}{2}}}{i} = \sqrt{\frac{2\varrho}{\pi}}\, \frac{ie^{-i\varrho}}{\varrho}.$$

Hence according to (15) we have

$$\zeta^1_0 = \frac{e^{i\varrho}}{i\varrho}, \qquad \zeta^2_0 = \frac{e^{-i\varrho}}{-i\varrho}. \tag{15a}$$

Concerning the notation we make the following remark: our notation coincides with the original definition of the ψ_n in Heine's *Handbook of Spherical Harmonics* and with that used by the author in Frank-Mises. However it differs by a factor ϱ from that of other authors[11] who instead

[11] P. Debye, *Ann. Physik* **30** (1909), B. van der Pol and H. Bremmer, *Phil. Mag.* **24** (1937). Further references in G. N. Watson, Theory of Bessel Functions, p. 56.

of (11) write:

$$I_{n+\frac{1}{2}}(\varrho) = \sqrt{\frac{2}{\pi\varrho}}\,\psi_n(\varrho)\,, \tag{16}$$

which is sometimes convenient.

In analogy to the equations (14a) for the ψ_n we can express the ζ_n in *elementary form with the help of* $\exp(\pm i\varrho)$. This states the fact, which will be derived in Section D, that Hankel's asymptotic expansions break off in the case of half-indices.

The differential equation for $Z_{\pm\frac{1}{3}}(kr)$ assumes an unexpectedly elegant form if we replace the independent and the dependent variable[12] by

$$kr = \frac{2}{3}\varrho^{\frac{3}{2}}, \qquad Z_{\pm\frac{1}{3}}(kr) = \varrho^{-\frac{1}{2}}\,\Phi(\varrho)\,. \tag{17}$$

The functions ψ_n in (16) are denoted by S_n in acoustical engineering; the corresponding C_n would, in our terminology, have to be called a "Neumann function," since it is proportional to $\zeta_n^1 - \zeta_n^2$.

[12] The direct computation would lead to lengthy transformations. We avoid them, and at the same time recognize the generalizability of the relations (17) to (20), if we start from the conformal mapping

$$x + iy = f(\xi + i\eta), \qquad \Delta_{xy} = \frac{1}{|f'(\xi+i\eta)|^2}\Delta_{\xi\eta},$$

through which

$$\Delta_{xy}\,u + k^2 u = 0 \tag{1}$$

goes over into

$$\Delta_{\xi\eta}\,v + k^2\,|f'(\xi+i\eta)|^2\,v = 0\,; \tag{2}$$

where $u(x, y) \equiv v(\xi, \eta)$ (see equation (23.17) below). If in (2) we set f proportional to a power of $\xi + i\eta$, e.g.,

$$kf(\xi+i\eta) = \frac{2}{\mu}(\xi+i\eta)^{\mu/2}, \qquad \begin{cases} x + iy = re^{i\varphi} \\ \xi + i\eta = \varrho e^{i\psi}, \end{cases}$$

then we get

$$k\,|x+iy| = kr = \frac{2}{\mu}\,|\xi+i\eta|^{\mu/2} = \frac{2}{\mu}\varrho^{\mu/2}, \qquad \varphi = \frac{\mu}{2}\psi, \tag{3}$$

and

$$k\,|f'(\xi+i\eta)| = \varrho^{\mu/2}/\varrho\,.$$

The solution of (1)

$$u = I_{\mp 1/\mu}(kr)\,e^{i\varphi/\mu}$$

then goes over into the solution of (2)

$$v = I_{\mp 1/\mu}\left(\frac{2}{\mu}\varrho^{\mu/2}\right)e^{i\psi/2}; \tag{4}$$

It then becomes

$$\Phi''(\varrho) + \varrho\,\Phi(\varrho) = 0\,. \tag{18}$$

If we write its solutions as a series with undetermined coefficients starting with ϱ^0 and ϱ^1, we get

$$\begin{aligned}\Phi_0 &= 1 - 1\cdot\frac{\varrho^3}{3!} + 1\cdot 4\cdot\frac{\varrho^6}{6!} - 1\cdot 4\cdot 7\cdot\frac{\varrho^9}{9!} + \cdots,\\ \Phi_1 &= \varrho - 2\cdot\frac{\varrho^4}{4!} + 2\cdot 5\cdot\frac{\varrho^7}{7!} - 2\cdot 5\cdot 8\cdot\frac{\varrho^{10}}{10!} + \cdots\end{aligned} \tag{19}$$

Considering (19.34) we see that the first is proportional to $I_{-\frac{1}{3}}$, the second to $I_{+\frac{1}{3}}$, namely, that we have

$$\begin{aligned}\Phi_0(\varrho) &= 3^{-\frac{1}{3}}\,\Gamma\left(1-\frac{1}{3}\right)\varrho^{\frac{1}{2}}\, I_{-\frac{1}{3}}\left(\frac{2}{3}\,\varrho^{\frac{3}{2}}\right),\\ \Phi_1(\varrho) &= 3^{+\frac{1}{3}}\,\Gamma\left(1+\frac{1}{3}\right)\varrho^{\frac{1}{2}}\, I_{+\frac{1}{3}}\left(\frac{2}{3}\,\varrho^{\frac{3}{2}}\right).\end{aligned} \tag{20}$$

We shall meet the functions $I_{\pm\frac{1}{3}}$ again at the end of Section D.

and therefore

$$\frac{1}{\varrho}\frac{\partial}{\partial\varrho}\left(\varrho\,\frac{\partial v}{\partial\varrho}\right) + \frac{1}{\varrho^2}\frac{\partial^2 v}{\partial\varphi^2} + \varrho^{\mu-2}\, v = 0\,.$$

Substituting $v = \varrho^{-\frac{1}{2}}\,\Phi(\varrho)\,e^{i\varphi/2}$, here we get:

$$\frac{d^2\Phi}{d\varrho^2} + \varrho^{\mu-2}\,\Phi = 0, \tag{5}$$

with the easily verifiable series representations for its solutions:

$$\begin{aligned}\Phi_0 &= 1 - \frac{\varrho^\mu}{\mu(\mu-1)} + \frac{\varrho^{2\mu}}{\mu(\mu-1)\,2\mu\,(2\mu-1)} - \cdots,\\ \Phi_1 &= \varrho - \frac{\varrho^{\mu+1}}{(\mu+1)\mu} + \frac{\varrho^{2\mu+1}}{(\mu+1)\,\mu(2\mu+1)\,2\mu} - \cdots.\end{aligned} \tag{6}$$

We can relate these solutions to the solutions (4) for v; namely we have:

$$\left.\begin{matrix}\Phi_0\\ \Phi_1\end{matrix}\right\} = C_\mp\,\varrho^{\frac{1}{2}}\, I_{\mp 1/\mu}\left(\frac{2}{\mu}\,\varrho^{\mu/2}\right). \tag{7}$$

where $C_\mp$ are constant factors. Substituting the power series for I from (19.34), and comparing with (6) we get

$$C_- = \mu^{-1/\mu}\,\Gamma\left(1-\frac{1}{\mu}\right), \qquad C_+ = \mu^{1/\mu}\,\Gamma\left(1+\frac{1}{\mu}\right). \tag{7a}$$

For $\mu = 3$ the equations (5), (6), (7), (7a) go over into the equations (18), (19), (20) of the text.

For $\mu = 2$ equation (5) reduces to the differential equation for the trigonometric functions, and Φ_0, Φ_1 become $\cos\varrho$, $\sin\varrho$. For $\mu = 1$ our series representation breaks down, since then $\varrho = 0$ is a singular point of the differential equation (5).

D. Generalization of the Saddle-Point Method According to Debye

Although in later applications we shall in general apply only the asymptotic limiting value of the Bessel functions as determined at the end of §19, we wish to discuss here certain more general expansions due to Hankel, which progress according to negative powers of ϱ and in which the first term is the above mentioned asymptotic limiting value. Actually these series are *divergent*, being developments at an essential singularity, but they are frequently called *semi-convergent*. The first terms decrease rapidly, but from a certain term on they increase to infinity. We obviously must break off at that term in order to obtain approximation formulas.

The shortest way of obtaining these series is from the differential equation for the Hankel function, by substituting formal power series, and then computing the coefficients by setting the factor of each power equal to zero (this is obviously not completely rigorous). Considering (19.55) and (19.56) we write

$$H_n^{1,2}(\varrho) = \sqrt{\frac{2}{\pi\varrho}}\, e^{\pm i(\varrho-(n+\frac{1}{2})\pi/2)}\left(a_0 + \frac{a_1}{\varrho} + \frac{a_2}{\varrho^2} + \cdots + \frac{a_m}{\varrho^m} + \cdots\right) \tag{22}$$

and after dividing out the factor $\sqrt{2/\pi}\, \exp\{\pm i(\varrho-(n+\frac{1}{2})\pi/2)\}$ from the differential equation (19.11) we find the terms with $\varrho^{-m-\frac{3}{2}}$ to be

$$\begin{array}{lll} -a_{m+1} \mp 2i(m+\tfrac{1}{2})\,a_m & + (m+\tfrac{1}{2})(m-\tfrac{1}{2})\,a_{m-1}, \\ \qquad\qquad \pm i\,a_m & -(m-\tfrac{1}{2})\,a_{m-1}, \\ +a_{m+1} & -n^2 a_{m-1}. \end{array}$$

where the consecutive rows correspond to the consecutive terms

$$\frac{d^2 Z}{d\varrho^2}, \qquad \frac{1}{\varrho}\frac{dZ}{d\varrho}, \qquad \left(1-\frac{n^2}{\varrho^2}\right)Z$$

in (19.11). Summing the three rows we get the following *first order recursion formula*

$$\mp 2\,i\,m\,a_m = (n^2 - \{m-\tfrac{1}{2}\}^2)\,a_{m-1}. \tag{23}$$

Setting $a_0 = 1$ we get

$$\frac{a_1}{\varrho} = \frac{4n^2-1}{2^2(\mp 2i\varrho)}, \qquad \frac{a_2}{\varrho^2} = \frac{(4n^2-1)(4n^2-9)}{2^4\,2!\,(\mp 2i\varrho)^2}, \ldots \tag{24}$$

Using the symbol

$$(n,m) = \frac{(4n^2-1)(4n^2-9)\cdots(4n^2-\{2m-1\}^2)}{2^{2m}\,m!}, \quad (n,0) = 1 \tag{25}$$

which was introduced by Hankel, we get the general formula:

$$\frac{a_m}{\varrho^m} = \frac{(n,m)}{(\mp 2i\varrho)^m}. \tag{26}$$

The series in (22) for H^1 and H^2 then assume the final form:

$$H_n^1(\varrho) = \sqrt{\frac{2}{\pi\varrho}}\, e^{i(\varrho-(n+\frac{1}{2})\pi/2)} \sum_{m=0,1,2,\ldots} \frac{(n,m)}{(-2i\varrho)^m}, \tag{27}$$

$$H_n^2(\varrho) = \sqrt{\frac{2}{\pi\varrho}}\, e^{-i(\varrho-(n+\frac{1}{2})\pi/2)} \sum_{m=0,1,2,\ldots} \frac{(n,m)}{(+2i\varrho)^m}. \tag{28}$$

Taking half their sum we get:

$$\begin{aligned} I_n = {} & \sqrt{\frac{2}{\pi\varrho}} \cos(\varrho-(n+\tfrac{1}{2})\pi/2) \sum_{m=0,2,4,\ldots} (-1)^{\frac{m}{2}} \frac{(n,m)}{(2\varrho)^m} \\ & - \sqrt{\frac{2}{\pi\varrho}} \sin(\varrho-(n+\tfrac{1}{2})\pi/2) \sum_{m=1,3,5,\ldots} (-1)^{\frac{m-1}{2}} \frac{(n,m)}{(2\varrho)^m}. \end{aligned} \tag{29}$$

In exercise IV.5 we shall apply a similar method in order to determine the leading terms of the series (27), (28) (which here were borrowed from the saddle-point method) from Bessel's differential equation with large ϱ. This method does not include the normalizing factor which remains undetermined by the differential equation.

Extensive mathematical investigations about the domain of validity of such asymptotic series exist, starting with a great work of Poincaré,[13] which we cannot discuss here. Exceptions to the divergence are the series with half-integer index $n = \nu + \frac{1}{2}$, which, according to the definition of the symbol (m,n) break off with the ν-th term and represent the Bessel function in question *exactly*. We then obtain the elementary expressions for ζ_n, ψ_n which were given in Section C.

Our considerations so far are essentially restricted by the condition $n \ll \varrho$; they fail if n becomes infinite with ϱ. The latter is the case in all optical problems which are on the border line between geometrical optics (optics of very short wavelengths) and wave optics. It was in connection with the investigation of a problem of this type, namely that of the rainbow (radius of water droplet approximately equal to

[13] *Acta Math.* 8, 1886.

wavelength of light) that Debye[14] discovered his fundamental generalization of Hankel's asymptotic series. In order to understand its origin we first have to generalize the saddle-point method.

The exponent

$$f(w) = i\,[\varrho \cos w + n\,(w - \pi/2)] \tag{30}$$

in our representation (19.22) of the Hankel functions now depends on two large numbers ϱ and n. For convenience we take ϱ and n to be real and positive. Depending on whether n is smaller or larger than ϱ we set

(30 a) $\quad n = \varrho \cos \alpha \quad$ or $\quad$ (30b) $\quad n = \varrho \cosh \alpha;$

in addition we use the variable of integration

$$\beta = w - \pi/2. \tag{30 c}$$

as in Fig. 18.

a) *For* $n < \varrho$ we have

$$f(w) = F(\beta) = -i\,\varrho\,(\sin \beta - \beta \cos \alpha). \tag{31}$$

The saddle point $F'(\beta) = 0$ is given by

$$\cos \beta - \cos \alpha = 0;$$

it lies at $\beta_0 = \mp\,\alpha$, for H^1 and H^2 respectively. This corresponds to the previous values for the saddle points $w_0 = 0,\ \pi$, which by (30c) go into $\beta_0 = \mp\,\pi/2$. From (31) we get

$$F''(\beta_0) = \mp\, i\,\varrho \sin \alpha$$

which yields as the expansion of $F(\beta)$ up to the quadratic term

$$F(\beta) = \pm\, i\,\varrho\,(\sin \alpha - \alpha \cos \alpha) \mp \frac{1}{2}\, i\,\varrho \sin \alpha\,(\beta - \beta_0)^2. \tag{31 a}$$

Instead of β we introduce the arc length s measured from the saddle point $\beta_0 = \mp\,\alpha$ and set $(\beta - \beta_0)^2 = (\beta \pm \alpha)^2 = \mp\, i s^2,\ d\beta = e^{\mp i\pi/4}\, ds,$ Concerning the $\mp$- sign in the last equation we refer the reader to the discussion in (19.54b). Integrating over a neighborhood of the saddle point we get

$$H_n^{1,2}(\varrho) = \frac{1}{\pi} \int e^{F(\beta)}\, d\beta = e^{\pm i\varrho(\sin \alpha - \alpha \cos \alpha)}\, \frac{1}{\pi} \int_{-s}^{+s} e^{-\frac{\varrho}{2} \sin \alpha \cdot s^2}\, e^{\mp \frac{i\pi}{4}}\, ds\,. \tag{31 b}$$

[14] *Math. Ann.* 67, 1909 and *Bayr. Akad.* 1910.

This again can be reduced to the Laplace integral. We have:

$$H_n^{1,2}(\varrho) = \sqrt{\frac{2}{\pi\varrho\sin\alpha}}\, e^{\pm i\varrho(\sin\alpha - \alpha\cos\alpha) \mp i\pi/4}. \tag{32}$$

In the limit $\alpha \to \pi/2$ our form (32) goes into the previous representation (19.55), (19.56).

b) The same calculation holds in the *case* $n > \varrho$ if in (30b) we replace $\cos\alpha$ by $\cosh\alpha$ and hence (31) by

$$F(\beta) = -i\varrho(\sin\beta - \beta\cosh\alpha)$$

That one of the two saddle points $\beta_0 = \pm i\alpha$, which yields the dominant term, is the one with greater altitude, namely $\beta_0 = -i\alpha$. At this point $F''(\beta_0) = \varrho \sinh\alpha$. Instead of (32) we now get

$$H_n^{1,2}(\varrho) = \sqrt{\frac{2}{\pi\varrho\sinh\alpha}}\, e^{\varrho(\alpha\cosh\alpha - \sinh\alpha) \mp i\pi/2}. \tag{33}$$

From these limiting values (32), (33) Debye deduced series developments of the Hankel type, which we may omit here.

c) The only remaining case is the transition case $n \sim \varrho$ in which according to (30a,b) we have $\alpha \sim 0$ and hence the representations (32), (33) fail on account of the denominators $\sqrt{\sin\alpha}$ and $\sqrt{\sinh\alpha}$. This indicates that now $F''(\beta_0)$, also approaches zero and that only the third term of the Taylor series for $F(\beta)$ is appreciably different from zero. We therefore need a better approximation in the neighborhood of the saddle point. This was carried out by Watson,[15] who instead of (31a) used an expansion which goes up to the third order in $(\beta - \beta_0)$. The Laplace integrals of the Airy type (see end of this section) which arise there can also be computed rigorously. We thus find:
in the case $n \leq \varrho$,

$$H_n^{1,2}(\varrho) = \frac{\tan\alpha}{\sqrt{3}}\, e^{\pm in(\tan\alpha - \frac{1}{3}\tan^3\alpha - \alpha) \pm i\pi/6}\, H_{\frac{1}{3}}^{1,2}\left(\frac{1}{3}\, n\tan^3\alpha\right); \tag{34}$$

in the case $n > \varrho$ (where $n = \varrho \cosh\alpha$ as in (30b)),

$$H_n^{1,2}(\varrho) = \frac{\tanh\alpha}{\sqrt{3}}\, e^{-n(\tanh\alpha + \frac{1}{3}\tanh^3\alpha - \alpha) \mp 2i\pi/3}\, H_{\frac{1}{3}}^{2,1}\left(\pm\frac{in}{3}\tanh^3\alpha\right). \tag{35}$$

Taken together the Watson formulas (34) and (35) cover the entire

[15] Chap. VIII, and in particular p. 252 of his Theory of Bessel Functions, Cambridge 1922.

asymptotic range of the Bessel-Hankel functions including the border case $n \sim \varrho$, we are now treating. In this case we are in the neighborhood of $\alpha = 0$. Hence we may replace $H_{\frac{1}{3}}$ by its limiting value for small arguments, which according to (19.31) and (19.30), is (since we may neglect $I_{\frac{1}{3}}$ as compared to $I_{-\frac{1}{3}}$)

$$H_{\frac{1}{3}}^{1,2}(z) = \mp \frac{i}{\sin \pi/3} I_{-\frac{1}{3}}(z) .$$

Since in (34) we have $z = \frac{1}{3} n \tan^3 \alpha$ we get $I_{-\frac{1}{3}}$ proportional to $1/\tan \alpha$, which cancels with the factor $\tan \alpha$ on the right side of (34); hence after the necessary contractions we get from (34)

$$H_{\varrho}^{1,2}(\varrho) = \frac{2}{\Gamma(\frac{2}{3})} \left(\frac{2}{9\varrho}\right)^{\frac{1}{3}} e^{\mp i\pi/3} . \tag{36}$$

The same expression is obtained from (35). As the corresponding limiting value of I we get

$$I_{\varrho}(\varrho) = \frac{1}{\Gamma(\frac{2}{3})} \left(\frac{2}{9\varrho}\right)^{\frac{1}{3}} . \tag{37}$$

This coincides with the original results of Debye.

We also see that if n is not too near ϱ, equations (34), (35) coincide with the Debye formulas (32), (33). For, in this case we may substitute its Hankel limiting value (19.55,56) for the function $H_{\frac{1}{3}}$ (large argument and small index):

$$H_{\frac{1}{3}}^{1,2}\left(\frac{1}{3} n \tan^3\alpha\right) = \sqrt{\frac{2 \cdot 3}{\pi n \tan^3 \alpha}}\, e^{\pm \frac{in}{3} \tan^3 \alpha \mp i(\frac{1}{3} + \frac{1}{2})\pi/2} ,$$

whereby (34) simplifies to

$$H_n^{1,2}(\varrho) = \sqrt{\frac{2}{\pi n \tan \alpha}}\, e^{\pm in(\tan \alpha - \alpha) \mp i\pi/4} .$$

Due to $n = \varrho \cos \alpha$ this coincides with (32). In the same way one shows that (35) and (33) coincide.

Finally we have to consider the problem of the roots of the equations $H_n^{1,2}(\varrho) = 0$ for large n and ϱ. However, while up to now we assumed n and ϱ to be real, we now have to admit arbitrary complex values for n; we still may assume ϱ to be real on account of its physical meaning ($\varrho = k r$) . Concerning the parameter α, whose sign is undetermined in (30a), we agree that its real part is to be positive.

It would seem from (32), (33) that no roots could exist even for complex n, since the *exponential function* vanishes for no finite value of the exponent. However these representations were obtained (see p. 119 under b) by considering only that one of the two saddle points which has the *greater* altitude. If they are of *equal altitude* and if the required path of integration (leading from depression to depression) can be made to lead over both passes, then as sum of the two exponential expressions we get a *trigonometric function*, which makes the existence of roots possible.

Using the notations of equations (30) to (32) we represent the saddle points by $\beta = \mp\alpha$ and the corresponding exponential functions which appear in H by

$$e^{\pm i\varrho(\sin\alpha - \alpha\cos\alpha) \mp i\pi/4}$$

The altitude of the passes is determined by the real part of the exponent. Equal altitudes therefore mean equal real parts of the exponents, and since ϱ was assumed real, this means equal imaginary parts of $\pm(\sin\alpha - \alpha\cos\alpha)$, or in other words,

$$\text{(38)} \qquad \operatorname{Im}(\sin\alpha - \alpha\cos\alpha) = 0.$$

For small α this yields:

$$\operatorname{Im}(\alpha^3) = 0,$$

This means that α lies on one of three curves that pass through the origin and intersect there at angles of $\pi/3$, one being the real α-axis. The real axis remains a solution of (38) even for infinite α, while the other branches are continued into curves that are mirror images with respect to the imaginary α-axis.

Considering the path of integration, described in Fig. 19 for the Hankel function, we see that the path of integration for H^1 can be taken meaningfully over the two saddle points only if they lie on the branch of (38) which leads from the second to the fourth quadrants, i.e., if α has a positive real part whenever it has a negative imaginary part; hence $n = \varrho\cos\alpha$ (with real ϱ) has a positive imaginary part. On the other hand the path of integration for H^2 must lead from the third to the first quadrant, so that α has a positive, and n a negative, imaginary part.

Superimposing the contributions of both saddle points according to (32) we get the following representation for H^1

$$\text{(39)} \qquad \begin{aligned} H_n^1(\varrho) &= \sqrt{\frac{2}{\pi\varrho\sin\alpha}}\left(e^{i\varrho(\sin\alpha - \alpha\cos\alpha) - i\pi/4} - e^{-i\varrho(\sin\alpha - \alpha\cos\alpha) + i\pi/4}\right) \\ &= 2i\sqrt{\frac{2}{\pi\varrho\sin\alpha}}\sin\left[\varrho(\sin\alpha - \alpha\cos\alpha) - \pi/4\right]. \end{aligned}$$

From this we obtain the roots of the equations $H_n^1(\varrho) = 0$ directly: they are the roots of the following transcendental equation in α

(40) $\varrho(\sin\alpha - \alpha\cos\alpha) - \pi/4 = -m\pi, \qquad m = 1, 2, 3, \ldots,$

where we have to choose the negative sign on the right in order to satisfy our requirement that α be in the fourth quadrant.

For small α we obtain from (40)

$$\varrho\frac{\alpha^3}{3} = -(4m-1)\pi/4$$

and after the correct choice of the cube root of unity, we have

$$\alpha = \left[\frac{3\pi}{4\varrho}(4m-1)\right]^{\frac{1}{3}} e^{-i\pi/3}.$$

Now α is related to ϱ by equation (30a), which for small α, after we make the substitution $\cos\alpha = 1 - \frac{\alpha^2}{2}$, yields:

(41) $$n = \varrho + \frac{1}{2}\varrho^{\frac{1}{3}}\left[\frac{3\pi}{4}(4m-1)\right]^{\frac{2}{3}} e^{i\pi/3}.$$

The roots of $H_n^1(\varrho) = 0$ lie in the *positive-imaginary n-half-plane*, a fact that we shall apply later, and they are *infinite in number*. If we solve (41) with respect to ϱ then we get values in the *negative-imaginary ϱ- half-plane*. According to (41) n and ϱ are of the same order of magnitude, as assumed in the beginning. Hence (41) is the solution of the root problem in question.

We see that the saddle-point method is very general. It can be transferred from the treatment of the Hankel functions to that of arbitrary integrals of the form

(42) $$\int e^{F(w,\varrho,n,\ldots)}\,dw,$$

where F depends on several large numbers $\varrho, n, \ldots$ in addition to the variable of integration w, and where the path of integration W starts in a complex region in which $\lim \exp F(w, \ldots) = 0$ and leads to a similar region. In the integrals of the type (42) when the saddle point $F' = 0$ approaches a point $F'' = 0$ one encounters the same peculiarities that we encountered in the case of Hankel functions for the border line case $n \sim \varrho$. This is the case of the Airy diffraction theory of the rainbow. The phenomenon of the rainbow is in fact linked to the appearance of a turning point ($F'' = 0$) in the wave front, which in the asymptotic approximation coincides with the saddle point $F' = 0$. The calculation[16]

[16] W. Wirtinger, *Berichte des Naturw.-mediz. Vereins in Innsbruck* **23**, 97(1896); J. W. Nicholson, *Phil. Mag.* **18**, (1909).

of the "Airy integral" in question leads then to the functions $H_{\frac{1}{3}}$ or, what is the same thing, to the functions $I_{\pm\frac{1}{3}}$ just as in (34) and (35).

§ 22. Spherical Harmonics and Potential Theory

A. The Generating Function

The simplest approach to the theory of spherical harmonics is given by potential theory. We start from the so-called Newtonian potential $1/r$ and, after shifting the origin from $x = y = z = 0$ to (x_0, y_0, z_0), obtain

$$(1) \qquad \frac{1}{R} = \frac{1}{\sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}} = \frac{1}{\sqrt{r^2 - 2rr_0 \cos\vartheta + r_0^2}}.$$

The polar coordinates r, ϑ, φ have been chosen so that the polar axis $\vartheta = 0$ goes through the point (x_0, y_0, z_0). We then have $x_0 = y_0 = 0$, $z_0 = r_0$ and

$$(2) \qquad \begin{aligned} x &= r \sin\vartheta \cos\varphi, \\ y &= r \sin\vartheta \sin\varphi, \\ z &= r \cos\vartheta. \end{aligned}$$

We may expand (1) in ascending or descending powers of r depending on whether $r < r_0$ or $r > r_0$. If we denote the coefficient of the n-th ascending or descending power by P_n we have:

$$(3) \qquad \frac{1}{R} = \begin{cases} \dfrac{1}{r_0} \displaystyle\sum_{n=0}^{\infty} \left(\frac{r}{r_0}\right)^n P_n(\cos\vartheta) & r < r_0, \\ \dfrac{1}{r} \displaystyle\sum_{n=0}^{\infty} \left(\frac{r_0}{r}\right)^n P_n(\cos\vartheta) & r > r_0. \end{cases}$$

The P_n must be the same in both expansions since they must coincide when $r = r_0$ and $\vartheta \neq 0$. The point $r = r_0$, $\vartheta = 0$ is a singular point, the sphere $r = r_0$ playing the role here that is played by the circle of convergence of the Taylor series in the two-dimensional case.

The polynomials P_n defined by (3) are of n-th degree in $\cos\vartheta$. They are called *spherical harmonics* and we shall show that they coincide with the polynomials P_n which were introduced in §5. The function $1/R$ is called the *generating function* of spherical harmonics.

B. Differential and Difference Equation

First we want to find the differential equation of *spherical harmonics.* The fundamental equation of potential theory $\Delta u = 0$, which is satisfied by $1/R$, can be written in the form

$$\frac{1}{r}\frac{\partial^2(ru)}{\partial r^2}+\frac{1}{r^2\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial u}{\partial\vartheta}\right)+\frac{1}{r^2\sin^2\vartheta}\frac{\partial^2 u}{\partial\varphi^2}=0. \tag{4}$$

Since this equation must be satisfied by each term in the series (3), we obtain from the n-th term of the first line of (3) after dividing out the common factor r^{n-2}/r_0^{n+1},

$$n(n+1)P_n+\frac{1}{\sin\vartheta}\frac{d}{d\vartheta}\left(\sin\vartheta\frac{dP_n}{d\vartheta}\right)=0; \tag{5}$$

the same follows from the second line after factoring out r_0^n/r^{n+3}. We introduce the abbreviation

$$\cos\vartheta=\zeta=\frac{z}{r}$$

and note that

$$-\sin\vartheta\,d\vartheta=d\zeta,\quad \sin\vartheta\frac{d}{d\vartheta}=\sin^2\vartheta\frac{d}{\sin\vartheta\,d\vartheta}=-(1-\zeta^2)\frac{d}{d\zeta}.$$

Equation (5) can then be written

$$\frac{d}{d\zeta}\left\{(1-\zeta^2)\frac{dP_n}{d\zeta}\right\}+n(n+1)P_n=0, \tag{6}$$

or

$$\left\{(1-\zeta^2)\frac{d^2}{d\zeta^2}-2\zeta\frac{d}{d\zeta}+n(n+1)\right\}P_n=0. \tag{6a}$$

We consider (6) with n replaced by l and then, following the scheme of Green's theorem, we multiply by P_l and P_n and subtract:

$$\begin{aligned}&P_l\frac{d}{d\zeta}\left\{(1-\zeta^2)\frac{dP_n}{d\zeta}\right\}-P_n\frac{d}{d\zeta}\left\{(1-\zeta^2)\frac{dP_l}{d\zeta}\right\}\\ &=\frac{d}{d\zeta}\left\{(1-\zeta^2)\left(P_l\frac{dP_n}{d\zeta}-P_n\frac{dP_l}{d\zeta}\right)\right\}=\{l(l+1)-n(n+1)\}P_lP_n.\end{aligned} \tag{7}$$

The physical range of the variable ζ is from $\zeta=-1$, to $\zeta=+1$, $(\vartheta=0)$. Integrating over this range we get the condition of orthogonality for $l\neq n$

$$\int_{-1}^{+1}P_lP_n\,d\zeta=0, \tag{8}$$

since the integral of the second line of (7) with respect to ζ vanishes unless the P become singular for $\zeta = \pm 1$ which is excluded by equation (3).

We now show that our P_n satisfy the normalizing condition (5.7)

(9) $$P_n(1) = 1$$

For $\cos\vartheta = \pm 1$ we get from (1):

$$\frac{1}{|r \mp r_0|} = \begin{cases} \frac{1}{r_0} \sum (\pm 1)^n \left(\frac{r}{r_0}\right)^n & r < r_0, \\ \frac{1}{r} \sum (\pm 1)^n \left(\frac{r_0}{r}\right)^n & r > r_0. \end{cases}$$

Comparing this with (3) we even get the somewhat more general equation:

(9a) $$P_n(\pm 1) = (\pm 1)^n.$$

Now we saw in §5 that the P_n were uniquely determined by the orthogonality (8) and the normalization (9). Hence our present definition, with the help of a generating function, leads to the same functions P_n as did the method of least squares in §5. In particular we have the representation (5.8)

(10) $$P_n(\zeta) = \frac{1}{2^n\, n!} \frac{d^n}{d\zeta^n} (\zeta^2 - 1)^n$$

and as a result, according to (5.12)

(10a) $$\int P_n^2(\zeta)\, d\zeta = \frac{1}{n + \frac{1}{2}}.$$

The P_n are even or odd functions of ζ according as n is even or odd:

(10b) $$P_n(-\zeta) = (-1)^n P_n(\zeta).$$

In addition to a differential equation with respect to the variable ζ, our generating function yields a *difference equation with respect to the index n*. We rewrite, say, the first line of equation (3) with the abbreviation $\alpha = r/r_0$:

(11) $$\frac{1}{\sqrt{\alpha^2 - 2\alpha\zeta + 1}} = \Sigma\, \alpha^n P_n;$$

By logarithmic differentiation with respect to α we obtain

(11a) $$\frac{\zeta - \alpha}{\alpha^2 - 2\alpha\zeta + 1} = \frac{\Sigma\, n\, \alpha^{n-1} P_n}{\Sigma\, \alpha^n P_n}$$

and after cross-multiplication

$$(\zeta-\alpha)\,\Sigma\,\alpha^n P_n = (\alpha^2-2\,\alpha\,\zeta+1)\,\Sigma\,n\,\alpha^{n-1} P_n.$$

If we compare the coefficients of the same power of α on both sides, say those of α^n, we obtain

$$\zeta\,P_n - P_{n-1} = (n-1)\,P_{n-1} - 2\,\zeta\,n\,P_n + (n+1)\,P_{n+1},$$

or

(11 b) $$(n+1)\,P_{n+1} - (2\,n+1)\,\zeta\,P_n + n\,P_{n-1} = 0.$$

The same *recursion formula* is, of course, obtained from the second line of (3).

By the logarithmic differentiation of (11) with respect to ζ we obtained a mixed *differential difference equation*.
Instead of (11a) we now obtain

(11c) $$\frac{\alpha}{\alpha^2-2\,\alpha\,\zeta+1} = \frac{\Sigma\,\alpha^n P'_n}{\Sigma\,\alpha^n P_n}, \quad P'_n = \frac{dP_n(\zeta)}{d\zeta}$$

and after cross-multiplication we get, from the coefficient of α^{n+1},

(11d) $$P_n - P'_{n-1} + 2\,\zeta\,P'_n - P'_{n+1} = 0.$$

Multiplying this equation by $2n+1$ and adding twice the equation obtained from (11b) by differentiation with respect to ζ, we obtain

$$-(2\,n+1)\,P_n - P'_{n-1} + P'_{n+1} = 0.$$

We rewrite this as the *differential recursion formula:*

(11e) $$\frac{d}{d\zeta}(P_{n+1} - P_{n-1}) = (2\,n+1)\,P_n.$$

C. Associated Spherical Harmonics

The potential equation (4) suggests that in addition to the particular solutions

(12) $$u_n = \left\{\begin{matrix} r^n \\ r^{-n-1} \end{matrix}\right\} P_n(\cos\vartheta)$$

which depend only on r and ϑ we might consider also the particular solutions

(12a)
$$u_{nm} = \begin{Bmatrix} r^n \\ r^{-n-1} \end{Bmatrix} P_n^m(\cos\vartheta)\, e^{im\varphi}$$

which depend on r, ϑ and φ, by *associating* to P_n certain *spherical harmonics* P_n^m (where m is an integer assumed positive for the time being) defined by the differential equation

(13)
$$\frac{1}{\sin\vartheta}\frac{d}{d\vartheta}\left(\sin\vartheta\,\frac{dP_n^m}{d\vartheta}\right) + \left\{n(n+1) - \frac{m^2}{\sin^2\vartheta}\right\} P_n^m = 0,$$

which follows from (4). Written in analogy to (6) and (6a) we have:

(13a)
$$\frac{d}{d\zeta}\left\{(1-\zeta^2)\frac{dP_n^m}{d\zeta}\right\} + \left\{n(n+1) - \frac{m^2}{1-\zeta^2}\right\} P_n^m = 0,$$

(13b)
$$\left\{(1-\zeta^2)\frac{d^2}{d\zeta^2} - 2\zeta\frac{d}{d\zeta} + n(n+1) - \frac{m^2}{1-\zeta^2}\right\} P_n^m = 0.$$

According to Thomson and Tait our original P_n are called *zonal-spherical harmonics* and the associated ones are called *tesseral.* The lines of zeros of the former divide the surface of the sphere into latitudinal regions of different signs, those of the latter divide it into quadrangles (tesserae) of different signs which are bounded by lines of latitude and of longitude. The associated or tesseral spherical harmonics are *orthogonal* for different lower but *equal upper* indices; namely as in (7) we conclude from our differential equation, which now is (13a), that

(14)
$$\int_{-1}^{+1} P_l^m P_n^m\, d\zeta = 0 \qquad \text{for} \qquad l \neq n.$$

In order to obtain an analytic expression for P_n^m we expand at the points $\zeta = \pm 1$ (north and south poles of the unit sphere) in powers of $\zeta \mp 1$, :

$$P_n^m = (\zeta \mp 1)^\lambda\,[a_0 + a_1(\zeta \mp 1) + a_2(\zeta \mp 1)^2 + \cdots].$$

This is analogous to (19.36). The determination of λ in analogy to (19.37), is obtained from the differential equation (13b):

(15)
$$\lambda(\lambda - 1) + \lambda - \frac{m^2}{4} = 0, \qquad \lambda = +\frac{m}{2}.$$

(The other root $\lambda = -m/2$ must be excluded for reasons of continuity.) We unite the branches at each of the points $\zeta = \pm 1$ into

$$(1-\zeta^2)^{m/2}$$

and write

$$P_n^m = (1-\zeta^2)^{m/2}\, v = \sin^m \vartheta \cdot v. \tag{16}$$

For the v which we introduced here we obtain from (13b)

$$\left\{(1-\zeta^2)\frac{d^2}{d\zeta^2} - 2(m+1)\,\zeta\frac{d}{d\zeta} + [n(n+1) - m(m+1)]\right\} v = 0, \tag{17}$$

which now must be solved in terms of series which contain only *integral* powers of $\zeta \mp 1$. However we do not have to investigate these series since the required integral of (17) can be obtained in closed form from (6a). Namely if we differentiate (6a) m times with respect to ζ and apply the well known rule of differentiation

$$\frac{d^m}{d\zeta^m}\zeta\frac{d}{d\zeta} = \left\{\zeta\frac{d}{d\zeta} + \binom{m}{1}\right\}\frac{d^m}{d\zeta^m}, \tag{17a}$$

then we obtain exactly the expression $\{\}$ of (17) applied to the m-th derivative of P_n. Hence we see that we obtain a solution of (17) by setting

$$v = \frac{d^m P_n}{d\zeta^m} \tag{17b}$$

With the use of (16) and (10) we obtain a simple representation for our associated spherical harmonics and at the same time a determination of the normalizing factor which has been free up to this point:

$$P_n^m = \frac{(1-\zeta^2)^{m/2}}{2^n\, n!}\;\frac{d^{n+m}(\zeta^2-1)^n}{d\zeta^{n+m}}. \tag{18}$$

Hence for even m, P_n^m is, like P_n, a polynomial of degree n; for odd m, P_n^m is $\sqrt{1-\zeta^2}$ times a polynomial of degree $n-1$. We further see from (18) that

$$P_n^0 = P_n, \qquad P_n^m = 0 \quad \text{for} \quad m > n. \tag{18a}$$

The last statement follows from the fact that for $m > n$ the order of differentiation in (18) is greater than the degree of the differentiated polynomial.

D. On Associated Harmonics with Negative Index m

Up to now we had to assume a positive index m, for example, in (17b) we made use of differentiation of order m with respect to ζ. However our final representation (18) can be extended directly to *negative* m with $m \geq -n$. We therefore extend (18) to the $2n + 1$ values $|m| \leq n$. For negative m too, the function P_n^m is a polynomial of degree n (in the same sense as for positive m). This is because the pole at $\zeta = \pm 1$ given by the factor $(1 - \zeta^2)^{m/2}$ for negative m is cancelled by the second factor of (18), which has a zero there of corresponding order due to the fact that the order of differentiation has been lowered by $|m|$. In addition the P_n^m satisfy the differential equation (13) for negative m too (since (18) depends only on m^2). Hence the P_n^m for negative m can differ from the $P_n^{|m|}$ only by a constant factor C, which is best determined by the comparison of the highest powers of ζ in P_n^{-m} and P_n^{+m} as calculated from (18). We then have:

$$\frac{P_n^{-m}}{P_n^{+m}} = (1-\zeta^2)^{-m} \frac{d^{n-m}}{d\zeta^{n-m}} \zeta^{2n} \Big/ \frac{d^{n+m}}{d\zeta^{n+m}} \zeta^{2n}$$

$$\sim (-1)^m \zeta^{-2m} \frac{(2n)!}{(n+m)!} \zeta^{n+m} \Big/ \frac{(2n)!}{(n-m)!} \zeta^{n-m} = (-1)^m \frac{(n-m)!}{(n+m)!};$$

hence:

$$P_n^{-m} = C \cdot P_n^{+m}, \qquad C = (-1)^m \frac{(n-m)!}{(n+m)!}. \tag{18b}$$

This equation holds for both positive and negative m.

Our definition of the P_n^m, which departs from the older mathematical literature, has been justified by wave mechanics and will also serve to unify our expressions.[17]

Hence we have exactly $2n + 1$ adjoined P_n^m, one of which coincides with P_n, the rest being pairwise equal except for the constant factor C; another difference is in the factor $\exp(i\, m\, \varphi)$ by which they are multiplied in (12a).

[17] In the older literature the upper index of P_n^m is assumed positive throughout and the φ-dependence is given by $\cos m\varphi$ or $\sin m\varphi$. It is much simpler to assume this dependence exponential, as we have done in (12a), where we also dropped the restriction to positive m.

An even greater departure from the customary definition is suggested by C. G. Darwin (*Proc. Roy. Soc. London* **115**, 1927) who appends the factor $(n - m)!$ to the right side of (18). Then (18b) simplifies to

$$P_n^m = (-1)^m P_n^{|m|} \quad \text{for} \quad m < 0.$$

But this definition implies a change in the classical expression for the Legendre polynomials $P_n = P_n^0$, which we want to avoid.

Moreover, some authors, in particular E. W. Hobson in his Theory of Spherical and Ellipsoidal Harmonics, Cambridge 1931, use the factor $(-1)^m$ in the definition of P_n^m in (18), but this is immaterial for our purposes.

E. Surface Spherical Harmonics and the Representation of Arbitrary Functions

By the most general "*surface spherical harmonic*" (introduced by Maxwell) we mean the expression

$$Y_n = \sum_{m=-n}^{+n} A_m P_n^m (\cos \vartheta) e^{im\varphi}. \tag{19}$$

which contains $2n + 1$ arbitrary constants. Multiplied by r^n (or r^{-n-1}) Y_n yields the most general potential of order n (or $-n-1$) which is *homogeneous* in the coordinates x,y,z (Maxwell's solid harmonics). It is a combination of the special u_{nm} of (12a) which also are homogeneous in x,y,z:

$$\left.\begin{matrix} r^n \\ r^{-n-1} \end{matrix}\right\} Y_n = \sum_{m=-n}^{+n} A_m u_{nm}, \tag{19a}$$

and the general non-homogeneous solution of the potential equation (4) is represented as a sum of its homogeneous parts:

$$u = \sum_{n=0}^{\infty} \left\{\begin{matrix} r^n \\ r^{-n-1} \end{matrix}\right\} Y_n. \tag{19b}$$

By restricting this representation to the case of a sphere of radius 1 and giving the value of u on the surface as $f(\varphi, \vartheta)$ we obtain

$$f(\vartheta, \varphi) = \sum_{n=0}^{\infty} Y_n = \sum_{n=0}^{\infty} \sum_{m=-n}^{+n} A_{nm} P_n^m (\cos \vartheta) e^{im\varphi}. \tag{20}$$

By using the notation A_{nm} instead of the A_m of (19) we emphasize the fact that the free constants in each Y_n are independent of and different from the constants in any other Y_n. The series (19b) and (20) express the fact that the *boundary value problem of potential theory for a sphere* is solvable for an arbitrary value $f(\vartheta, \varphi)$ of the potential on the surface of the sphere, both for the interior of the sphere (factor r^n in (19b)) and for the exterior (factor r^{-n-1}). In the following section we shall treat this problem by direct construction of Green's function and thereby derive the above series again. The first rigorous proof of (20) under very general assumptions on the nature of $f(\vartheta, \varphi)$ was given by Dirichlet in 1837.

F. Integral Representation of Spherical Harmonics

We now consider a special homogeneous function of degree n in x,y,z

(21) $(z + i\,x)^n = r^n(\cos\vartheta + i\sin\vartheta\cos\varphi)^n = r^n(\zeta + \sqrt{\zeta^2 - 1}\cos\varphi)^n$,

which, like every function of the form $f(z + ix)$ or $f(z + iy)$ etc., obviously satisfies the equation $\Delta u = 0$. Hence the coefficient of r^n in (21) is a surface spherical harmonic Y_n. If we average it over φ, making it a pure function of ζ, we obtain our zonal spherical harmonic

$$P_n(\zeta) = \frac{1}{2\pi}\int_{-\pi}^{+\pi}(\zeta + \sqrt{\zeta^2 - 1}\cos\varphi)^n\,d\varphi\,. \tag{22}$$

If on the other hand we construct the m-th Fourier coefficient of Y_n as in (1.12) then we obtain the associated (tesseral) spherical harmonic

$$P_n^m(\zeta) = \frac{C}{2\pi}\int_{-\pi}^{+\pi}(\zeta + \sqrt{\zeta^2 - 1}\cos\varphi)^n e^{-im\varphi}\,d\varphi\,. \tag{23}$$

The integral representation (22) is first mentioned by Laplace in his Mécanique Céleste, Vol. V. The fact that the denominator 2π provides the correct normalization is seen by setting $\zeta = 1$, for then the integrand becomes 1 and hence $P_n(1) = 1$. On the other hand we still have to determine the normalizing factor C in (23). By comparison with the normalization of (18) we find[18]

$$C = \frac{(n+m)!}{n!}\,e^{-im\pi/2}. \tag{23a}$$

If in (21) we replace n by $-n-1$, as we know is possible, we get, equivalent to (22), the representation

$$P_n(\zeta) = \frac{1}{2\pi}\int_{-\pi}^{+\pi}\frac{d\varphi}{(\zeta + \sqrt{\zeta^2 - 1}\cos\varphi)^{n+1}}\,, \tag{23b}$$

which is also seen to be normalized.

[18] For instance by passing to the limit $\zeta \to \infty$ in both (23) and (18), whereby, except for the factors ζ^n and $\exp(-im\varphi)$, the integrand in (23) reduces to

$$(1 + \cos\varphi)^n = 2^n\cos^{2n}\varphi/2 = 2^{-n}e^{in\varphi}(1 + e^{-i\varphi})^{2n}.$$

In the binomial expansion of this expression we have to consider only the term with $e^{im\varphi}$ since all the other terms vanish upon integration with respect to φ. The factor $\exp\{-im\pi/2\}$ in (23a) is due to the factor $\sin^m\vartheta$ in (18). See also the similar passage to the limit $\zeta \to \infty$ in (18b).

G. A Recursion Formula for the Associated Harmonics

Starting from the recursion formulas (11b) and (11e) for the zonal spherical harmonics, we differentiate (11b) m times with respect to ζ, apply rule (17a) to the middle term and multiply each term by $\sin^m \vartheta$. As a result of (18) we obtain

$$(24)\quad (n+1)\,P^m_{n+1} - (2n+1)\,\zeta\,P^m_n - m\,(2n+1)\sin\vartheta\,P^{m-1}_n + n\,P^m_{n-1} = 0.$$

On the other hand we obtain from (11e) upon $(m-1)$-fold differentiation with respect to ζ and multiplication by $\sin^m \vartheta$:

$$(25)\qquad P^m_{n+1} - P^m_{n-1} = (2n+1)\sin\vartheta\,P^{m-1}_n.$$

Eliminating the term with $\sin\vartheta$ from (24) and (25) we obtain the recursion formula

$$(26)\quad (n+1-m)\,P^m_{n+1} - (2n+1)\,\zeta\,P^m_n + (n+m)\,P^m_{n-1} = 0.$$

which is a generalization of (11b). Apparently this equation holds only for positive m, due to its derivation through m-fold differentiation. However we can verify it for negative m if we consider our general definition (18) of P^m_n and the relation (18b).

H. On the Normalization of Associated Harmonics

From (10a) we know the value of the normalizing integral for $m = 0$. We denote it by N_n or also by N^0_n. Its computation in §5 is based on the symbol $D_{k,l}$ of (5.9). We first consider the generalized form of the normalizing integral

$$(27)\qquad N^{\pm m}_n = \int_{-1}^{+1} P^m_n(\zeta)\,P^{-m}_n(\zeta)\,d\zeta.$$

Written in terms of the symbol $D_{k,l}$ we have as a result of our general definition (18)

$$(28)\qquad N^{\pm m}_n = \frac{1}{2^{2n}\,n!\,n!}\int_{-1}^{+1} D_{n+m,n}\cdot D_{n-m,n}\,d\zeta.$$

Through m-fold integration by parts we obtain, since the terms outside the sign of integration vanish for $\zeta = \pm 1$:

$$(29)\qquad N^{\pm m}_n = \frac{(-1)^m}{2^{2n}\,n!\,n!}\int_{-1}^{+1} D_{n,n}\cdot D_{n,n}\,d\zeta = (-1)^m\,N^0_n = \frac{(-1)^m}{n+\frac{1}{2}}.$$

Here in the last equation we have substituted the value of N_n° from (10a). The normalizing integral is usually taken as

(30)
$$N_n^m = \int_{-1}^{+1} P_n^m(\zeta)\, P_n^m(\zeta)\, d\zeta .$$

This can be deduced directly from (29) by using the relation (18b) which yields

(31)
$$N_n^m = \frac{1}{C} N_n^{\pm m} = \frac{1}{n+\frac{1}{2}} \frac{(n+m)!}{(n-m)!} .$$

Its direct computation in the manner of (28) would have been somewhat more cumbersome.

We remark that in the following chapter we shall always carry out a "normalization of the eigenfunction to 1." If we denote the associated harmonic with this normalization by Π_n^m, then we have:

(31 a)
$$\int_{-1}^{+1} \Pi_n^m(\zeta)\, \Pi_n^m(\zeta) d\zeta = 1 ,$$

and comparing this with (30) we get

(31 b)
$$\Pi_n^m = P_n^m / \sqrt{N_n^m} = P_n^m \cdot \left[(n+\tfrac{1}{2}) \frac{(n-m)!}{(n+m)!} \right]^{\frac{1}{2}} .$$

J. The Addition Theorem of Spherical Harmonics

The proof of this theorem is based on a lemma, which we shall be able to prove only with the methods of the following chapter and which shall be assumed here without proof, namely: *The surface spherical harmonic*

(32)
$$Y_n = \sum_{m=-n}^{+n} \Pi_n^m (\cos\vartheta)\, e^{im\varphi} \cdot \Pi_n^m (\cos\vartheta_0)\, e^{-im\varphi_0}$$

depends only on the relative position of the two points (ϑ, φ) *and* (ϑ_0, φ_0) *on the surface of the sphere,* in other words it has an invariant meaning independent of the coordinate system. If we now change the coordinate system of ϑ, φ by letting the polar axis of a new coordinate system Θ, Φ pass through the point (ϑ_0, φ_0), then the latter has the coordinate $\Theta_0 = 0$ (its Φ_0 becomes undetermined), while the Θ coordinate of the former point (ϑ, φ) is now given by

(33)
$$\cos\Theta = \cos\vartheta \cos\vartheta_0 + \sin\vartheta \sin\vartheta_0 \cos(\varphi - \varphi_0)$$

For $\Theta_0 = 0$ all the terms in (32) vanish except those with $m = 0$. Hence the right side of (32) becomes the product of the zonal spherical harmonics $\Pi_n(\cos\Theta)$ and $\Pi_n(1)$. Due to the stated invariance of Y_n we then have

$$(34) \qquad \Pi_n(\cos\Theta)\,\Pi_n(1) = \sum_{m=-n}^{+n} \Pi_n^m(\cos\vartheta)\,\Pi_n^m(\cos\vartheta_0)\,e^{im(\varphi-\varphi_0)}.$$

This is the *symmetric form* of the addition theorem which expresses its structure in a convincingly simple form. The form which is common in the literature is obtained by expressing the Π_n^m in terms of P_n^m with the help of (31b) and (31). Equation (34) then becomes

$$(35) \qquad P_n(\cos\Theta) = \sum_{m=-n}^{+n} \frac{(n-m)!}{(n+m)!} P_n^m(\cos\vartheta)\,P_n^m(\cos\vartheta_0)\,e^{im(\varphi-\varphi_0)},$$

or written in real form

$$(36) \qquad \begin{aligned} P_n(\cos\Theta) &= P_n(\cos\vartheta)\,P_n(\cos\vartheta_0) \\ &\quad + 2\sum_{m=1}^{n} \frac{(n-m)!}{(n+m)!} P_n^m(\cos\vartheta)\,P_n^m(\cos\vartheta_0)\cos m(\varphi-\varphi_0). \end{aligned}$$

It is however evident that the true structure of the addition theorem is gradually lost in the passage (34) → (35) → (36).

Another rather transparent form of the addition theorem is obtained from (35) by replacing one of the upper indices m by $-m$ and applying (18b):

$$(37) \qquad P_n(\cos\Theta) = \sum_{m=-n}^{+n} (-1)^m P_n^m(\cos\vartheta)\,P_n^{-m}(\cos\vartheta_0)\,e^{im(\varphi-\varphi_0)}.$$

§ 23. Green's Function of Potential Theory for the Sphere. Sphere and Circle Problems for Other Differential Equations

We superimpose two principal solutions u, u' of the potential equation $\Delta u = 0$,

$$(1) \qquad \begin{aligned} u &= \frac{e}{R}, \quad R^2 = (x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2, \\ u' &= \frac{e'}{R'}, \quad R'^2 = (x-\xi')^2 + (y-\eta')^2 + (z-\zeta')^2 \end{aligned}$$

and seek the level surfaces of the function $G = u - u'$, in particular the surface $G = 0$. According to (1) the latter is given by the equation $R'^2 = (e'/e)^2 R^2$, or written explicitly:

$$\begin{aligned}(2)\qquad &\left(1-\frac{e'^2}{e^2}\right)(x^2+y^2+z^2)\\ &-2\left\{\left(\xi'-\frac{e'^2}{e^2}\xi\right)x+\left(\eta'-\frac{e'^2}{e^2}\eta\right)y+\left(\zeta'-\frac{e'^2}{e^2}\zeta\right)z\right\}\\ &+\xi'^2-\frac{e'^2}{e^2}\xi^2+\eta'^2-\frac{e'^2}{e^2}\eta^2+\zeta'^2-\frac{e'^2}{e^2}\zeta^2=0.\end{aligned}$$

This is the equation of a *sphere*. The position of its center and the length of its radius can be calculated from (2): the center O lies on the connecting line of the "source points" $Q = (\xi, \eta, \zeta)$ and $Q' = (\xi', \eta', \zeta')$, the radius a is obtained as the *mean proportional* of $OQ = \varrho$ and $OQ' = \varrho'$,

$$(3)\qquad \varrho\,\varrho' = a^2,$$

so that one of the source points lies in the interior, the other in the exterior of the sphere of radius a.

For further discussion we shall not use the cumbersome formula (2), but rather the elementary geometric Fig. 22.

A. Geometry of Reciprocal Radii

Fig. 22 illustrates the method of *reciprocal radii*[19] formulated in (3). We call Q' the *inverse image* of Q with respect to the sphere of radius a, or also the *electric image* (Maxwell); the notations e and e' in (1) are connected with the electric point of view. The relation between Q and Q' is symmetric: Q is the inverse image of Q'. From (3) we see in the well known manner that the points Q, Q' are harmonic with respect to the points of intersection P_1, P_2 of the line OQQ' and the sphere.

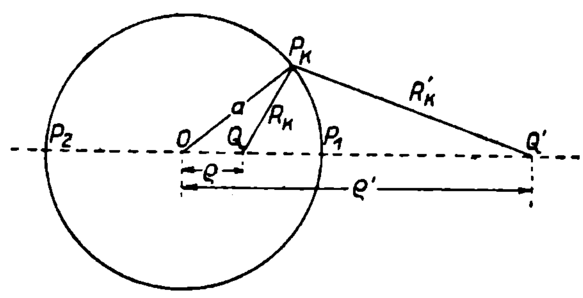


Fig. 22. Geometry of reciprocal radii. Q and Q' are transformed into each other by inversion on the sphere of radius a around the center O. The triangles OQP_k and OP_kQ' are similar.

The *method of reciprocal radii* was developed through the pioneering

[19] The term "*reciprocal*" arises from the (bad) habit of setting $a = 1$, in which case $\varrho' = 1/\varrho$. For reasons of dimensionality we consider it better to retain the radius a as a length.

work of William Thomson[20] who applied it to a wide range of problems in electro- and magnetostatics. Transformation by reciprocal radii is called *inversion* for short.

From (3) it follows that the triangle OP_kQ of the figure is similar to the triangle $OQ'P_k$, hence we have:

$$\frac{\varrho}{R_k} = \frac{a}{R'_k}, \quad \text{where } R_k = P_kQ \text{ and } R'_k = P_kQ'. \tag{4}$$

Here P_k denotes a special point on the surface of the sphere while the symbols P,R,R' are reserved for an arbitrary point P: (x,y,z) and its distances from Q and Q'. In order to determine the "image charge" e' which was introduced in (1) we compare (4) with the relation

$$\frac{e}{R_k} = \frac{e'}{R'_k}$$

which follows from (1), and is valid for every point P_k. We obtain

$$\frac{e'}{e} = \frac{a}{\varrho} = \frac{\varrho'}{a} \qquad \text{(the latter due to equation (3)).} \tag{5}$$

B. The Boundary Value Problem of Potential Theory for the Sphere, the Poisson Integral

Equation (5) brings us back to our starting point, the condition $G = u - u' = 0$; and we can now justify the notation G, which signifies *Green's function.* We have in fact

$$G = G(P,Q) = \frac{e}{R} - \frac{e'}{R'} \tag{6}$$

as Green's function for the "interior boundary value problem" which is: to find a potential U which has no singularities in the interior of the sphere for a given boundary value $\overline{U}$ on the surface of the sphere. In the same manner

$$G = G(P,Q') = \frac{e'}{R'} - \frac{e}{R} \tag{6 a}$$

is Green's function for the corresponding "exterior boundary value problem." Of the three conditions a),b),c) (see p. 50) that serve to define Green's function, we have b) satisfied on account of (5) and a) satisfied because the potential equation is self-adjoint and therefore the

[20] *Journal de Math.* **10** (1845), **12** (1847). Maxwell in his Treatise, vol. I. Chap. XI, quotes a paper in the *Cambridge and Dublin Math. Journ.* of 1848.

differential equation of G coincides with that of U. In order to satisfy condition c) of the unit source we merely have to make

$$e \text{ or } e' = -1/4\pi .$$

for the inner or outer boundary value problem, as seen from the table on p. 49.

We write (6) explicitly by introducing the spherical coordinates r, ϑ, φ for P($r = 0$ is the center of the sphere, $\vartheta = 0$ an arbitrary direction). Let the corresponding coordinates for Q and Q' be:

$$r_0,\ \vartheta_0,\ \varphi_0 \quad \text{with} \quad r_0 = \varrho,$$
$$r_0',\ \vartheta_0',\ \varphi_0' \quad \text{with} \quad r_0' = \varrho',\ \vartheta_0' = \vartheta_0,\ \varphi_0' = \varphi_0;$$

as in (22.33) we let

$$\cos\Theta = \cos\vartheta\cos\vartheta_0 + \sin\vartheta\sin\vartheta_0\cos(\varphi - \varphi_0);$$

for e'/e we use the first of the values given by (5), and we let $e = -1/4\pi$. Equation (6) then becomes

$$(7)\qquad -4\pi G = \frac{1}{\sqrt{r^2 + \varrho^2 - 2r\varrho\cos\Theta}} - \frac{a/\varrho}{\sqrt{r^2 + \frac{a^4}{\varrho^2} - 2r\frac{a^2}{\varrho}\cos\Theta}}.$$

The solution for the interior boundary value problem according to the scheme of (10.12) is then

$$(8)\qquad U(Q) = \int \overline{U}\frac{\partial G}{\partial n}\, d\sigma .$$

The integration on the right is with respect to the point P and is taken over the surface of the sphere, $\partial G/\partial n = \partial G/\partial r$ for $r = a$ and $d\sigma = a^2\sin\vartheta\, d\vartheta\, d\varphi$; Q is an arbitrary point in the interior of the sphere. From (7) we get the general relation

$$4\pi\frac{\partial G}{\partial r} = \frac{r - \varrho\cos\Theta}{R^3} - \frac{a}{\varrho}\,\frac{r - \frac{a^2}{\varrho}\cos\Theta}{R'^3},$$

where R and R' are as before. Hence, for the surface of the sphere, where according to (4) we have $R_k' = \frac{a}{\varrho}R_k$, we get

$$4\pi\frac{\partial G}{\partial r} = \frac{1}{R_k^3}\left\{a - \varrho\cos\Theta - \frac{\varrho^2}{a^2}\left(a - \frac{a^2}{\varrho}\cos\Theta\right)\right\} = \frac{a}{R_k^3}\left(1 - \frac{\varrho^2}{a^2}\right).$$

Therefore, if we set $\overline{U} = f(\vartheta, \varphi)$ equation (8) becomes

$$(9)\qquad 4\pi U(Q) = a^3\left(1 - \frac{\varrho^2}{a^2}\right)\iint \frac{f(\vartheta,\varphi)\sin\vartheta\, d\vartheta\, d\varphi}{(a^2 + \varrho^2 - 2a\varrho\cos\Theta)^{\frac{3}{2}}}.$$

This representation was deduced by Poisson in a very circuitous manner through the development of $f(\vartheta, \varphi)$ in spherical harmonics. Here we see that the direct way is through Green's function (7).

The corresponding formula for the outer boundary value problem is obtained from (6a) by setting $e' = -1/4\pi$ and taking the second value in (5) for e'/e. We have:

$$(9a) \qquad 4\pi U(Q') = a^3\left(\frac{\varrho'^2}{a^2} - 1\right)\iint \frac{f(\vartheta, \varphi)\sin\vartheta\, d\vartheta\, d\varphi}{(a^2 + \varrho'^2 - 2a\varrho'\cos\Theta)^{\frac{3}{2}}}.$$

The so-called "second boundary value problem," in which we set $\partial G/\partial n = 0$ on the surface of the sphere, can not be solved with the method of reciprocal radii.

We now wish to gain a clearer geometrical understanding of the way in which formula (9); which is analytic throughout, can, on the surface, represent an arbitrary function $f(\vartheta, \varphi)$, which is in general not analytic. For this purpose we have to consider the passage to the limit $Q \rightarrow K$ as $\varrho \rightarrow a$ In this limit the factor $1-\varrho^2/a^2$ in front of the integral in (9) vanishes and hence only those elements of area $d\sigma$ contribute to the integral for which the denominator R^3 vanishes. The latter approaches zero for $\varrho \rightarrow a$ only when $\cos\Theta = 1$, and therefore $\vartheta = \vartheta_0$, $\varphi = \varphi_0$. Thus the only determining part is a neighborhood of that element of area which approaches Q, in other words the special value $f(\vartheta_0, \varphi_0)$ on this element of area will alone determine the limiting value. We indicate this fact by rewriting (9) in the form

$$(10) \quad 4\pi \lim_{Q \rightarrow K} U(Q) = a^3\left(1 - \frac{\varrho^2}{a^2}\right) f(\vartheta_0, \varphi_0) \int_0^{\varepsilon}\int_0^{2\pi} \frac{\sin\Theta\, d\Theta\, d\Phi}{(a^2 + \varrho^2 - 2a\varrho\cos\Theta)^{\frac{3}{2}}}.$$

In the numerator of the integrand we are allowed to replace $\sin\vartheta\, d\vartheta\, d\varphi$ by $\sin\Theta\, d\Theta\, d\Phi$; after this is done the integration can be carried out explicitly. We obtain

$$\frac{2\pi}{a\varrho}\left(\frac{1}{a-\varrho} - \frac{1}{(a^2 + \varrho^2 - 2a\varrho\cos\varepsilon)^{\frac{1}{2}}}\right).$$

Substituting this in (10) we observe that the contribution of the second term in the parentheses vanishes for $\varrho \rightarrow a$. From the first term, after dividing the denominator $a-\varrho$ into the factor $1-\varrho^2/a^2$, we get the desired value:

$$\lim_{Q \rightarrow K} U = f(\vartheta_0, \varphi_0).$$

In the two-dimensional case (circle instead of sphere) we can carry out a simplified consideration in close analogy to (9) and (9a), where instead of (9) we get

$$2\pi U(Q) = a^2\left(1 - \frac{\varrho^2}{a^2}\right)\int \frac{f(\varphi)\, d\varphi}{a^2 + \varrho^2 - 2a\varrho\cos(\varphi - \varphi_0)}\,. \tag{11}$$

C. General Remarks about Transformations by Reciprocal Radii

Returning to the three-dimensional case we now wish to consider the transformation by reciprocal radii from more general viewpoints. We choose an arbitrary point 0 to be the *center of inversion* and, at the same time, the origin of a spherical polar coordinate system; we then select a sphere with a center 0 and an arbitrary radius a as the *sphere of inversion.* An arbitrary point P: (r, ϑ, φ) is transformed into a point P': (r', ϑ', φ'). Between these points we have the relations:

$$\begin{aligned} r\,r' &= a^2, \qquad & \vartheta' &= \vartheta, \qquad & \varphi' &= \varphi; \\ dr &= -\frac{a^2}{r'^2}\,dr', \qquad & d\vartheta &= d\vartheta', \qquad & d\varphi &= d\varphi'. \end{aligned} \tag{12}$$

For the sake of completeness we also give the corresponding relations between the rectangular coordinates x,y,z and x',y',z'. Using the scheme of (22.2) we obtain from (12)

$$x' = r'\sin\vartheta'\cos\varphi' = \frac{a^2}{r}\sin\vartheta\cos\varphi = \frac{a^2}{r^2}\,x, \text{ etc.}$$

or, written in summarized form,

$$(x', y', z') = \frac{a^2}{r^2}\,(x, y, z)\,; \tag{12a}$$

and conversely

$$(x, y, z) = \frac{a^2}{r'^2}\,(x', y', z')\,. \tag{12b}$$

We now seek the transformation in polar coordinates of the line element ds^2 into ds'^2. According to (12) we have

$$\begin{aligned} ds^2 &= dr^2 + r^2\,d\vartheta^2 + r^2\sin^2\vartheta\,d\varphi^2 \\ &= \left(\frac{a}{r'}\right)^4 (dr'^2 + r'^2\,d\vartheta'^2 + r'^2\sin^2\vartheta'\,d\varphi'^2) = \left(\frac{a}{r'}\right)^4 ds'^2. \end{aligned} \tag{13}$$

Hence the transformations by reciprocal radii are *conformal:* every infinitesimal triangle with sides ds is transformed into a *similar* triangle with sides ds' (ratio of sides $=(a/r')^2=(r/a)^2$). According to a theorem by Liouville these mappings are the only non-trivial conformal transformations in three-dimensional space.

The geometric characterization of our transformation consists of the fact that it transforms spheres into spheres, where the plane has to be considered as a sphere of infinite radius. A *sphere* that passes through the center of inversion is transformed into a *plane*, since the center of inversion is transformed into infinity. (Infinity in this "geometry of spheres" is a *point* and not a *plane* as it is in projective geometry.) Conversely, a plane that does not pass through the center of inversion is transformed into a sphere.

D. Spherical Inversion in Potential Theory

The next point of interest to us is the transformation of the differential parameter Δu: we start from a function $u(r, \vartheta, \varphi)$ and transform the *product* ru by reciprocal radii (r = distance from center of inversion). We denote the new function by

$$v(r', \vartheta', \varphi') = \frac{a^2}{r'} u\left(\frac{a^2}{r'}, \vartheta', \varphi'\right). \tag{14}$$

In other words, we transfer the value ru from the original point P (r, ϑ, φ) to the point P' with the coordinates

$$r' = \frac{a^2}{r}, \quad \vartheta' = \vartheta, \quad \varphi' = \varphi \tag{14 a}$$

and we want to show that the differential parameter $\Delta' v$ which is calculated in terms of the coordinates r', ϑ', φ' is given by

$$\Delta' v = \left(\frac{a}{r'}\right)^4 r\, \Delta u. \tag{15}$$

Again, the reason for this relation lies in the conformality of the mapping, as indicated by the appearance in (15) of the square of the ratio of dilation $(r/a)^2$ from equation (13). Equation (15) can be proven as follows: we define the operators Δ and D as in (22.4) by the formulas

$$r^2 \Delta = \frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right) + D, \tag{15 a}$$

$$D = \frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta \frac{\partial}{\partial\vartheta}\right) + \frac{1}{\sin^2\vartheta}\frac{\partial^2}{\partial\varphi^2}. \tag{15 b}$$

Then, if Δ' and D' stand for the same expressions in r', ϑ', φ' , we obtain from (14) and (14a)

(15 c) $$D' v = r\, Du, \quad \frac{\partial v}{\partial r'} = \frac{\partial (r u)}{\partial r} \frac{dr}{dr'} = -\frac{a^2}{r'^2} \frac{\partial (r u)}{\partial r},$$

(15 d) $$\frac{\partial}{\partial r'}\left(r'^2 \frac{\partial v}{\partial r'}\right) = -a^2 \frac{\partial^2 (r u)}{\partial r^2} \frac{dr}{dr'} = \frac{a^4}{r'^2} \frac{\partial^2 (r u)}{\partial r^2};$$

and hence according to (15a,c,d)

(15 e) $$r'^2 \Delta' v = \frac{a^4}{r'^2}\left(\frac{\partial^2 (r u)}{\partial r^2} + \frac{r\, r'^2}{a^4} Du\right) = \frac{a^4}{r'^2}\left(\frac{\partial^2 (r u)}{\partial r^2} + \frac{1}{r} Du\right).$$

According to equation (22.4) the expression in the last parentheses is just $r\,\Delta u$. Hence (15e) becomes

(16) $$\Delta' v = \left(\frac{a}{r'}\right)^4 r\, \Delta u,$$

which coincides with (15).[21]

If we start from a function u which satisfies the differential equation $\Delta u = 0$ *in the coordinates x,y,z, then the function* $v = ru$ *after transformation by reciprocal radii satisfies the differential equation* $\Delta v = 0$ *in the coordinates x′,y′,z′.*

This theorem (William Thomson) enables us to transfer solutions of potential problems obtained for a certain region of space S to the transformed region S'. In particular this holds for Green's function: if it is known for a region S bounded by planes with the boundary condition $G = 0$, then our theorem gives Green's function G' for an arbitrary region S' bounded by spheres where the boundary condition $G' = 0$ remains valid. Depending on the position of the center of inversion, the region S' may have diverse shapes. The totality of those regions which were treated in §17 with the help of elementary reflections now becomes a richer manifold of regions bounded by spheres, thus permitting our generalized reflection by inversion. As before, this more general reflection leads to a simple and complete covering of space. The previous condition that all face angles must be submultiples of π remains valid owing to the conformality of the mapping. Where there was an infinity of image points (e.g., plane plate) there will still be an infinity of image points (e.g., the region between spheres tangent at the center of inversion, which is the image of the plane plate). Where there was a finite number of image points (e.g., for the wedge of 60° in Fig. 17), the inversion process for the spherical problem again terminates after a finite number of steps.

Examples will be given in exercises IV.6 and IV.7, where we shall also discuss the problem of a suitable choice of the center of inversion.

[21] Here the reason for the retention of a becomes apparent: if we had $a = 1$ the dimensionality of the factor $1/r'^4$ in (16) would not be understood, whereas now the dimensional consistency is clear.

Obviously all that has been said above can be transferred to two-dimensional potential theory, where inversion in a sphere becomes inversion in a circle. At the same time the range of possible mappings is increased tremendously since every transformation $z' = f(z)$ where f is an analytic function of the complex variable $z = x + iy$ leads to a conformal mapping. The dilation ratio of the line elements is then $|df/dz|$ and (16) is replaced by

$$\Delta' v = \left|\frac{df}{dz}\right|^2 \Delta u . \tag{17}$$

E. The Breakdown of Spherical Inversion for the Wave Equation

Unfortunately, these mapping methods for the two- and three-dimensional case are *entirely restricted to potential theory.* If we were to perform a transformation by reciprocal radii on the wave equation

$$\Delta u + k^2 u = 0 \tag{18}$$

then according to (16) the factor $(a/r')^4 r$ would appear and (18) would become:

$$\Delta' v + k^2 \left(\frac{a}{r'}\right)^4 v = 0 . \tag{19}$$

Only in the potential equation ($k = 0$) does this disturbing factor $(a/r')^4$ disappear. In the wave equation this factor means that the originally homogeneous medium (k constant) appears transformed into a highly inhomogeneous medium, which, in the neighborhood of the point $r' = 0$, shows a lens-like singularity of the index of refraction. The same holds for the equation of heat conduction, which, written in our customary form with u as temperature and k as temperature conductivity, would go into

$$\Delta' v = \frac{1}{k}\left(\frac{a}{r'}\right)^4 \frac{\partial v}{\partial t} . \tag{19a}$$

This form of the equation certainly can not serve to simplify the boundary value problem for the sphere. Instead, we have to rely on the much more cumbersome method of series expansion as applied in the corresponding two-dimensional case of §20 A.

§ 24. More About Spherical Harmonics

A. The Plane Wave and the Spherical Wave in Space

The simplest solution of the three-dimensional wave equation

$$\Delta u + k^2 u = 0 \tag{1}$$

is the plane wave, e.g., a purely periodic sound wave which progresses in the z-direction

$$u = e^{ikz} = e^{i\varrho\cos\vartheta}, \quad \varrho = kr, \quad k = \text{ wave number.} \tag{2}$$

If we develop this solution in zonal spherical harmonics $P_n(\cos\vartheta)$ then the coefficients will be the $\psi_n(\varrho)$ of §21 C. This follows from the wave equation on the one hand, and the differential equation of the P_n on the other hand. Using the left side of (22.4) and the postulated independence from φ, equation (1) becomes

$$\frac{1}{r}\frac{\partial^2 ru}{\partial r^2} + \frac{1}{r^2}\frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}\sin\vartheta\frac{\partial u}{\partial\vartheta} + k^2 u = 0\,.$$

Hence u can be separated into a product of P_n by a function $R(r)$ which depends only on r. Due to the differential equation (22.5) of P_n the function R must satisfy the equation

$$\frac{1}{r}\frac{d^2 rR}{dr^2} + \left(k^2 - \frac{n(n+1)}{r^2}\right)R = 0$$

which, in terms of the ϱ of equation (2), can be rewritten as:

$$\frac{1}{\varrho}\frac{d^2\varrho R}{d\varrho^2} + \left(1 - \frac{n(n+1)}{\varrho^2}\right)R = 0\,. \tag{3}$$

This is the same differential equation as (21.11a); the solutions, which were continuous for $\varrho = 0$, were defined as ψ_n. Neglecting a multiplicative constant we get from (21.11):

$$R(r) = \psi_n(\varrho) = \sqrt{\frac{\pi}{2\varrho}}\, I_{n+\frac{1}{2}}(\varrho)\,. \tag{4}$$

Similarly we obtain the linear combinations of the $\zeta^{1,2}(\varrho)$, defined in (21.15), as solutions of (3) discontinuous for $\varrho = 0$. Since the latter do not enter into the expansion of the plane wave, we have to write:

$$e^{i\varrho\cos\vartheta} = \sum_{n=0}^{\infty} c_n\,\psi_n(\varrho)\,P_n(\cos\vartheta)\,. \tag{5}$$

Here the coefficients c_n are still undetermined. They are determined from the orthogonality of the P_n. Namely, according to (22.8) and (22.10a) we get, if we again denote the variable of integration by $\zeta = \cos\vartheta$:

(6) $$c_n \psi_n(\varrho) = (n + \tfrac{1}{2}) \int_{-1}^{+1} e^{i\varrho\zeta} P_n(\zeta)\, d\zeta .$$

We now compare the asymptotic values for $\varrho \to \infty$ of the two sides. Due to the relation of ψ_n to $I_{n+\frac{1}{2}}$, we get for the left side from equation (19.57)

(6a) $$c_n \frac{\cos[\varrho - (n+1)\pi/2]}{\varrho} .$$

The integral on the right side can be expanded into a series in $1/\varrho$ through successive integrations by parts. Ignoring all higher powers of $1/\varrho$ for this integral we obtain:

(6b) $$\frac{e^{i\varrho}}{i\varrho} P_n(1) - \frac{e^{-i\varrho}}{i\varrho} P_n(-1) = \frac{1}{i\varrho}[e^{i\varrho} - (-1)^n e^{-i\varrho}] = 2 i^n \frac{\sin(\varrho - n\pi/2)}{\varrho} .$$

The coefficient of $2i^n$ here is the same as the coefficient of c_n in (6a). Substituting (6a,b) in (6) we therefore get

(6c) $$c_n = (2n + 1)\, i^n .$$

Hence the expansion (5) of the plane wave assumes the final form

(7) $$e^{i\varrho\cos\vartheta} = \sum_{n=0}^{\infty} (2n+1)\, i^n \psi_n(\varrho)\, P_n(\cos\vartheta) .$$

This should be compared with the Fourier expansion (21.2b) of the two-dimensional plane wave. Just as we considered the latter as generating function of the I_n, so we may consider the three-dimensional plane wave as the *generating function of the* ψ_n. At the same time (6) and (6c) yield the following integral representation of the ψ_n :

(7a) $$2 i^n \psi_n(\varrho) = \int_{-1}^{+1} e^{i\varrho\zeta} P_n(\zeta)\, d\zeta ,$$

The next simple solution of the wave equation (1) is the *spherical wave*

(8) $$u = \frac{e^{ikr}}{ikr} = \frac{e^{i\varrho}}{i\varrho} .$$

This represents a *radiated* wave which progresses in the positive r-

direction if we give its time dependence by $\exp(-i\omega t)$. According to (21.15a) the solution (8) is identical with the solution of (3):

$$\zeta_0^1 = \sqrt{\frac{\pi}{2\varrho}}\, H_{\frac{1}{2}}^1(\varrho). \tag{8a}$$

which is singular at the point $r = 0$. We now transfer the source point $r = 0$ to the arbitrary point

$$Q = (r_0, \vartheta_0, \varphi_0)$$

Then (8) becomes

$$u = \frac{e^{ikR}}{ikR} = \zeta_0^1(kR) \begin{cases} R = \sqrt{r^2 + r_0^2 - 2\, r\, r_0 \cos\Theta}, \\ \cos\Theta = \cos\vartheta\cos\vartheta_0 + \sin\vartheta\sin\vartheta_0\cos(\varphi - \varphi_0). \end{cases} \tag{8b}$$

This function too can be expanded in spherical harmonics $P_n(\cos\Theta)$. Here the coefficients must again be solutions of the differential equation (3), namely,

$$\psi_n(\varrho) \quad \text{for} \quad r < r_0, \quad \zeta_n^1(\varrho) \quad \text{for} \quad r > r_0,$$

the former, since the point $r = 0$ is now a regular point of the spherical wave, the latter, since the type of the *radiated* wave must be preserved in every term of the expansion. Owing to the symmetry of R in r and r_0 the reverse holds for the dependence on r_0. Hence in the coefficients of $P_n(\cos\Theta)$ we must have the factors

$$\zeta_n^1(\varrho_0) \quad \text{for} \quad r < r_0, \quad \psi_n(\varrho_0) \quad \text{for} \quad r > r_0,$$

so that the expansion reads

$$\frac{e^{ikR}}{ikR} = \begin{cases} \sum\limits_{n=0}^{\infty} c_n\, \zeta_n^1(\varrho_0)\, \psi_n(\varrho)\, P_n(\cos\Theta) & r < r_0, \\ \sum\limits_{n=0}^{\infty} c_n\, \psi_n(\varrho_0)\, \zeta_n^1(\varrho)\, P_n(\cos\Theta) & r > r_0. \end{cases} \tag{9}$$

The numerical factors c_n must be the same in both rows, since for $r = r_0$ the two rows coincide (except for the point Q, where we have $\Theta = 0$ and both series diverge). The situation here is the same as for the cylindrical wave in §21, equation (4): in the interior of the sphere we have a "Taylor series," in the exterior a series of the "Laurent type." The c_n can again be determined by passing to the limit $r \to \infty$. We get

$$R = r\left(1 - \frac{r_0}{r}\cos\Theta + \cdots\right) \to r - r_0\cos\Theta,$$

$$e^{ikR} \to e^{ikr}\, e^{-ikr_0\cos\Theta}.$$

Using (7) with $-i\varrho_0 \cos\Theta$ instead of $+i\varrho\cos\vartheta$ the left side of (9) becomes

$$\frac{e^{i\varrho}}{i\varrho}\sum(2n+1)(-i)^n\psi_n(\varrho_0)P_n(\cos\Theta).$$

Due to (21.15) and (19.55) the second line on the right side of (9) becomes in the limit

$$\sum c_n\psi_n(\varrho_0)\sqrt{\frac{\pi}{2\varrho}}\sqrt{\frac{2}{\pi\varrho}}\,e^{i[\varrho-(n+1)\pi/2]}P_n(\cos\Theta).$$

This will correspond term for term with the left side if we set

(9a) $$c_n = 2n+1.$$

We may also consider this representation of the spherical wave as an *addition theorem for the function*

$$\zeta_0^1(kR) = \zeta_0^1\left(\sqrt{\varrho^2+\varrho_0^2-2\varrho\varrho_0\cos\Theta}\right).$$

If, on the left side of (9), we pass from the radiated to the absorbed spherical wave

$$\frac{e^{-ikR}}{-ikR} = \zeta_0^2\left(\sqrt{\varrho^2+\varrho_0^2-2\varrho\varrho_0\cos\Theta}\right)$$

then throughout the right side ζ^2 must be replaced by ζ^1. From half the sum of both representations we obtain the *addition theorem* for the regular "*standing wave*"

(10) $$\psi_0(kR) = \frac{\sin kR}{kR} = \sum(2n+1)\psi_n(\varrho_0)\psi_n(\varrho)P_n(\cos\Theta),$$

here the distinction between $r \lesseqgtr r_0$ is unnecessary.

B. Asymptotic Behavior

If in the differential equation (22.13) of the associated spherical harmonics we pass to the limit

(11) $$n\to\infty,\quad \vartheta\to 0,\quad n\vartheta\to\eta,\quad P_n^m(\cos\vartheta)\to O_m(\eta),$$

then we obtain

(11a) $$\frac{1}{\eta}\frac{d}{d\eta}\left(\eta\frac{dO_m}{d\eta}\right)+\left(1-\frac{m^2}{\eta^2}\right)O_m = 0.$$

This is the differential equation (19.11) of the cylindrical harmonic Z_m.

Since P_n^m and hence O_m is finite for $\vartheta \to 0$, the only permissible solution of (11a) is the Bessel function I_m. Hence we have

$$O_m(\eta) = C_m I_m(\eta) \quad \text{with} \quad C_0 = 1. \tag{12}$$

The latter follows from the fact that for $m = 0$ and $\eta = 0$ we have $I_0(\eta) = 1$ on one hand and (due to (11)) $\vartheta = 0$ on the other, and hence $P_n(\cos\vartheta) = 1$ and $O_0(\eta) = 1$. In order to determine C_m for $m > 0$ too, we use (22.18), which for $\vartheta \to 0$ yields:

$$P_n^m \to \frac{\vartheta^m}{2^n n!} \lim_{\zeta \to 1} \frac{d^{n+m}}{d\zeta^{n+m}} (\zeta - 1)^n (\zeta + 1)^n. \tag{12a}$$

We rewrite the function under differentiation in the form

$$(\zeta - 1)^n 2^n \left(1 + \frac{\zeta - 1}{2}\right)^n = \cdots + 2^n (\zeta - 1)^n \binom{n}{m} \left(\frac{\zeta - 1}{2}\right)^m + \cdots.$$

In this binomial expansion we have written only one term since the terms of lower degree vanish upon differentiation and those of higher degree vanish in the limit $\zeta \to 1$. The $(n + m)$-fold differentiation of this term yields

$$2^{n-m} (n + m)! \binom{n}{m} = 2^{n-m} \frac{n!}{m!} \frac{(n + m)!}{(n - m)!}.$$

Upon substitution in (12a) we obtain

$$P_n^m \to \frac{1}{m!} \left(\frac{\vartheta}{2}\right)^m \frac{(n + m)!}{(n - m)!}. \tag{12b}$$

Here the last fraction has $2m$ more factors in its numerator than in its denominator; since $m \ll n$, we may identify all these factors with n to obtain

$$P_n^m \to \frac{1}{m!} \left(\frac{n\vartheta}{2}\right)^m n^m = \frac{1}{m!} \left(\frac{\eta}{2}\right)^m n^m. \tag{12c}$$

Comparing this with (12), where we replace $I_m(\eta)$ by the first term of its power series (19.34), we obtain

$$C_m = n^m. \tag{13}$$

Hence for $m > 0$ we must, in order to obtain I_m, divide P_n^m by n^m before passing to the limit.

The geometrical meaning of our result is as follows: The surface of the sphere can be replaced by its tangent plane for the neighborhood of the north pole $\vartheta \to 0$. The solution of the spatial wave equation, whose behavior on the sphere is determined by $P_n^m e^{im\varphi}$, thereby goes into a solution of the wave equation for the tangent plane, namely,

$I_m(\eta)\, e^{im\varphi}$, provided we perform the passage to the limit on P_n^m/n^m instead of P_n^m. The same obviously also holds for the south pole of the sphere $\vartheta \to \pi$.

Having thus treated the special cases $\vartheta \to 0$ and $\vartheta \to \pi$ we now wish to investigate the asymptotic value of P_n^m as $n \to \infty$ for a general $0 < \vartheta < \pi$. To this end we apply the saddle-point method to the integral (22.23), which we rewrite in the following complex form:

$$(14) \quad P_n^m(\zeta) = \frac{C}{2\pi i}\oint e^{n f(w)}\, dw, \qquad \text{with } w = e^{i\varphi}, \quad C = \frac{(n+m)!}{n!}\, e^{-i\, m\pi/2},$$

the latter due to (22.23a). The integration is to be taken over the unit circle of the w-plane in the positive (counterclockwise) sense; the function $f(w)$ stands for

$$(15) \qquad f(w) = \log\left\{\cos\vartheta + \frac{i}{2}\sin\vartheta\cdot(w + 1/w)\right\} - \frac{m+1}{n}\log w.$$

Hence

$$f'(w) = \frac{\frac{i}{2}\sin\vartheta\,(1 - 1/w^2)}{\cos\vartheta + \frac{i}{2}\sin\vartheta\cdot(w + 1/w)} - \frac{m+1}{nw}.$$

We therefore have two saddle points w_0, which, for $m \ll n$ and $\sin\vartheta \neq 0$, lie on the unit circle, namely,

$$w_0 = \pm 1$$

and we get

$$(15a) \qquad f''(w_0) = \sin\vartheta\, e^{\mp i(\vartheta - \pi/2)}.$$

With the same assumptions we get

$$(15b) \qquad e^{n f(w_0)} = e^{\pm i n\vartheta}\,(\pm 1)^{m+1}.$$

As in (19.54) we set for the two saddle points $w \mp 1 = s\, e^{i\gamma}$ and after applying (15a) we obtain

$$(15c) \quad f(w) - f(w_0) = f''(w_0)\,\frac{(w \mp 1)^2}{2} + \ldots = \frac{s^2}{2}\sin\vartheta\, e^{2i\gamma \mp i(\vartheta - \pi/2)}.$$

If we let

$$(15d) \quad 2i\gamma \mp i\,(\vartheta - \pi/2) = \pm i\pi \quad \text{and hence} \quad \gamma = \pm(\vartheta/2 + \pi/4),$$

then $f(w) - f(w_0)$ becomes real and $= -\frac{s^2}{2}\sin\vartheta$. This choice of γ means that for the saddle points we shall integrate along the line of steepest descent whose direction, according to (15d), still depends on ϑ. The two integrals then assume the common value

$$\int_{-s}^{+s} e^{-\frac{n s^2}{2}\sin\vartheta}\, ds, \tag{16}$$

Due to (15d) and the relation $d w = e^{i\gamma} ds$ this must be multiplied by the factor

$$e^{i\gamma} = e^{\pm i(\vartheta/2 + \pi/4)} \tag{16a}$$

In the limit $n \to \infty$ the integral (16) can be reduced to the Laplace integral by a simple substitution and we obtain

$$\sqrt{\frac{2\pi}{n \sin\vartheta}}. \tag{16b}$$

Due to (15c) and (16a,b) equation (14) becomes

$$P_n^m = \frac{C}{2\pi i}\sqrt{\frac{2\pi}{n\sin\vartheta}}\left(e^{i((n+\frac{1}{2})\vartheta + \pi/4)} + e^{-i((n+\frac{1}{2})\vartheta + \pi/4 + (m+1)\pi)}\right). \tag{16c}$$

Since we have made the assumption $m \ll n$ throughout, the value of C in (14) can be reduced to $C = n^m \exp\{-i m \pi/2\}$ by the same reasoning that led from (12b) to (13). Hence C/i becomes

$$n^m \exp\{-i(m+1)\pi/2\}$$

Combining this with the two exponential functions in (16c) we obtain:

$$P_n^m = n^m \sqrt{\frac{2}{\pi n \sin\vartheta}} \cos\left\{(n+\tfrac{1}{2})\vartheta - \frac{m\pi}{2} - \frac{\pi}{4}\right\}. \tag{17}$$

Therefore P_n^m for real n is a rapidly oscillating function of varying amplitude; the amplitude is small in the neighborhood of $\vartheta = \pi/2$ and increases symmetrically for decreasing or increasing ϑ. For $\vartheta = 0$ and π equation (17) breaks down since, according to (15a), $f''(w_0)$ vanishes and the series for $f(w) - f(w_0)$ starts with the third term (compare with the limiting case on p. 122 that led to the Airy integral). Equation (17) is then replaced by (12c).

We shall apply (17) in the appendix of Chapter VI for the case of complex n with a positive real part in which our derivation remains valid.

C. The Spherical Harmonic as an Electric Multipole

In this section we return to potential theory. Since in §22 E we were able to define the surface spherical harmonics of degree n as homogeneous potentials of degree n (or better of degree $-n-1$), it must be possible to generate them with the help of repeated differentiation "with respect to n-directions" of the elementary potential $1/R$. This is the point of view of Maxwell in Chapter IX of his treatise. We express this by the Maxwell rule:

$$(18)\quad Y_n = \frac{1}{n!}\frac{\partial}{\partial h_1}\frac{\partial}{\partial h_2}\cdots\frac{\partial}{\partial h_n}\left(\frac{1}{R}\right)\cdots \quad \begin{cases} R^2 = (\xi - x)^2 + (\eta - y)^2 + (\zeta - z)^2, \\ \text{Lim } x,\ y,\ z \to 0, \quad \text{Lim } R \to r, \\ r^2 = \xi^2 + \eta^2 + \zeta^2 = 1. \end{cases}$$

The "action point" $P = (\xi, \eta, \zeta)$ is to lie on a sphere of radius 1, the "source point" $Q = (x,y,z)$ is to lie in the neighborhood of the origin. The "directional differentiations" $h_1, h_2 \ldots, h_n$ can be performed both on the coordinates of P and on the coordinates of Q. We do the latter and then pass to the limit $x,y,z \to 0$, $R \to r$. In this way we obtain a *multipole* at Q whose order increases with the order of differentiation.

We start with the simplest case in which the directions $h_1, h_2 \ldots$ coincide, say, with the z-direction. The surface spherical harmonic which is obtained in this way is symmetric with respect to the z-axis and hence is a *zonal* spherical harmonic of the Legendre type P_n. We follow its genesis from line to line denoting the limit process of (18) by $\to$:

1) $$\frac{\partial}{\partial h_1}\frac{1}{R} = \frac{\partial}{\partial z}\frac{1}{R} = \frac{\zeta - z}{R^3} \to \zeta = P_1,$$

2) $$\frac{1}{2!}\frac{\partial}{\partial h_1}\frac{\partial}{\partial h_2}\frac{1}{R} = \frac{1}{2}\frac{\partial^2}{\partial z^2}\frac{1}{R} = \frac{1}{2}\frac{\partial}{\partial z}\frac{\zeta - z}{R^3} = -\frac{1}{2}\frac{1}{R^3} + \frac{3}{2}\frac{(\zeta - z)^2}{R^5}$$
$$\to \frac{3}{2}\zeta^2 - \frac{1}{2} = P_2,$$

3) $$\frac{1}{3!}\frac{\partial}{\partial h_1}\frac{\partial}{\partial h_2}\frac{\partial}{\partial h_3}\frac{1}{R} = \frac{1}{3!}\frac{\partial^3}{\partial z^3}\frac{1}{R} = \frac{1}{3}\frac{\partial}{\partial z}\left(-\frac{1}{2}\frac{1}{R^3} + \frac{3}{2}\frac{(\zeta - z)^2}{R^5}\right)$$
$$= -\frac{3}{2}\frac{\zeta - z}{R^5} + \frac{5}{2}\frac{(\zeta - z)^3}{R^7} \to \frac{5}{2}\zeta^3 - \frac{3}{2}\zeta = P_3,$$

4) $$\frac{1}{4!}\frac{\partial}{\partial h_1}\frac{\partial}{\partial h_2}\frac{\partial}{\partial h_3}\frac{\partial}{\partial h_4}\frac{1}{R} = \frac{1}{4!}\frac{\partial^4}{\partial z^4}\frac{1}{R} = \frac{1}{4}\frac{\partial}{\partial z}\left(-\frac{3}{2}\frac{\zeta - z}{R^5} + \frac{5}{2}\frac{(\zeta - z)^3}{R^7}\right)$$
$$= \frac{3}{8}\frac{1}{R^5} - \frac{15}{4}\frac{(\zeta - z)^2}{R^7} + \frac{35}{8}\frac{(\zeta - z)^4}{R^9} \to \frac{35}{8}\zeta^4 - \frac{15}{4}\zeta^2 + \frac{3}{8} = P_4.$$

This sequence $P_1, \ldots, P_4$, which can be completed by the zeroth derivative $P_0 = 1$ of $1/R$, coincides with the values obtained from the original definition on p. 23 (the variable x being replaced by ζ). This follows by necessity from the relation between spherical harmonics and homogeneous potentials, so that in (18) we were free to determine the normalizing factor $1/n!$ only. We note the connection between this rule and the second equation (22.3) which, after the substitution $r_0 = z$ (Q on the z-axis) and $r = 1$ (P on the unit sphere), can be written:

$$\frac{1}{R} = \sum_{m=0}^{\infty} z^m P_m(\cos\vartheta);$$

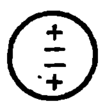
and hence for $z \to 0$ we indeed have

$$\frac{1}{n!}\frac{d^n}{dz^n}\frac{1}{R} = P_n(\cos\vartheta) = P_n(\zeta).$$

We list the names and symbols for the successive multipoles. In order to avoid the limit process $Q \to 0$ we replace the differentiation with respect to z (coordinate of source point) by a differentiation with respect to ζ (coordinate of action point) but with opposite sign, and interpret r as the distance $OP = \sqrt{\xi^2 + \eta^2 + \zeta^2}$.

Unipole charge scheme  potential $\frac{1}{r}$,

Bipole charge scheme  potential $-\frac{1}{1!}\frac{d}{d\zeta}\frac{1}{r}$,

Quadrupole charge scheme  potential $+\frac{1}{2!}\frac{d^2}{d\zeta^2}\frac{1}{r}$.

By contracting two quadrupoles of opposite scheme we obtain the "octupole" with potential

$$-\frac{1}{3!}\frac{d^3}{d\zeta^3}\frac{1}{r}.$$

(The determination of the corresponding charge scheme is left to the reader.) By n-fold differentiation we obtain the

$$2^n\text{-pole and its potential } \frac{(-1)^n}{n!}\frac{d^n}{d\zeta^n}\frac{1}{r}.$$

In wireless telegraphy one uses the term dipole instead of bipole. Quadrupole and octupole radiations occur in atomic physics.

We now have to consider examples of differentiations with respect to *different directions*. In addition to differentiations in the z-direction we now consider differentiations in the (x,y)-plane. In order to preserve a certain degree of symmetry we consider differentiation with respect to say, m equally spaced directions in the (x,y)-plane (in "star form," with an angle of π/m between two adjacent directions) together with $n-m$ differentiations still taken with respect to the z-direction. We thus obtain the *tesseral surface spherical harmonics*

(19) $$P_n^m(\zeta)\,\Phi_m(\varphi), \quad \Phi_m(\varphi) = e^{\pm i m \varphi},$$

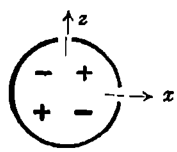
where instead of the two exponential functions given here we may have a linear combination, e.g., $\genfrac{}{}{0pt}{}{\cos}{\sin} m\varphi$. To this category (19) belong the so-called *sectorial surface spherical harmonics* (this notation too is Maxwell's) with $m = n$, which according to (22.18) is represented by

(19a) $$P_n^n \Phi_n = \sin^n \vartheta \frac{d^n P_n}{d\zeta^n} \Phi_n = \frac{(2n)!}{2^n n!} \sin^n \vartheta\, e^{\pm i n \varphi}.$$

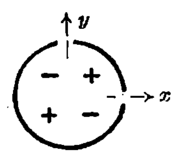
We discuss this further for $n = 2$. The star formed arrangement is obtained here if we take h_1 and h_2 in the x- and y-direction. Equation (18) and what follows then yield

$$P_2^2 = \frac{1}{2}\frac{\partial^2}{\partial x\,\partial y}\frac{1}{R} = \frac{1}{2}\frac{\partial}{\partial x}\frac{\eta - y}{R^3} = \frac{3}{2}\frac{(\xi - x)(\eta - y)}{R^5}$$
$$\rightarrow \frac{3}{2}\,\xi\,\eta = \frac{3}{4}\sin^2\vartheta \sin 2\varphi,$$

which is indeed of the type (19a). In this case too we speak of a *quadrupole* (see the right hand side of the diagram below; the left side, which is placed differently in space belongs to P_2^1).



Quadrupole P_2^1



Quadrupole P_2^2.

The fact that (18) yields the complete system of the $2n + 1$ surface spherical harmonics of degree n, follows from the number of constants in (18): two directional constants for every differentiation h and one multiplicative factor.

D. Some Remarks about the Hypergeometric Function

The hypergeometric function is best defined by its differential equation:

(20) $$z(1-z)\,y'' + [\gamma - (\alpha + \beta + 1)\,z]\,y' - \alpha\beta\,y = 0$$

From this equation we deduce the Gaussian series representation (11.10a) according to the procedure of §19 C. We set

(21) $$y = z^\lambda (a_0 + a_1 z + a_2 z^2 + \cdots + a_k z^k + \cdots),$$

with undetermined exponent λ and coefficients a_k. We substitute this in (20) and set the coefficients of the initial term $z^{\lambda-1}$ and the general term $z^{\lambda+k}$ equal to zero. In this way on one hand we obtain:

$$\lambda(\lambda - 1 + \gamma) = 0, \tag{21a}$$

and on the other hand

$$\begin{aligned}[(\lambda + k + 1)(\lambda + k) + \gamma(\lambda + k + 1)]\, a_{k+1} \\ = [(\lambda + k)(\lambda + k - 1) + (\alpha + \beta + 1)(\lambda + k) + \alpha\beta]\, a_k .\end{aligned} \tag{21b}$$

Equation (21a) has the solutions

$$\lambda = 0 \qquad \text{and} \qquad \lambda = 1 - \gamma; \tag{22a}$$

We first consider the former solution and by substituting it in (21b) obtain

$$a_{k+1} = \frac{k(k-1) + (\alpha+\beta+1)k + \alpha\beta}{(k+1)k + \gamma(k+1)}\, a_k = \frac{(\alpha+k)(\beta+k)}{(k+1)(\gamma+k)}\, a_k . \tag{22b}$$

Hence if we set $a_0 = 1$ we obtain the Gauss series

$$y = y_1 = F(\alpha, \beta, \gamma, z) = 1 + \frac{\alpha\beta}{1\cdot\gamma} z + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1\cdot 2\cdot\gamma(\gamma+1)} z^2 + \cdots . \tag{23}$$

The other solution of (22a) yields:

$$y = y_2 = z^{1-\gamma} F(\alpha - \gamma + 1,\ \beta - \gamma + 1,\ 2 - \gamma,\ z). \tag{23a}$$

There are also a large number of related representations for altered parameters α, β, γ and linearly transformed z, which coincide re-ion-wise with (23). They have been compiled lovingly by Gauss as "relationes inter contiguas."

If we compare the differential equation (22.6a) of the zonal spherical harmonics

$$(1 - \zeta^2) P'' - 2\zeta P' + n(n+1) P = 0 \tag{24}$$

with equation (20) then we see that it is obtained from the latter by the substitution

$$z = \frac{1 \mp \zeta}{2}, \qquad \alpha = -n, \qquad \beta = n + 1, \qquad \gamma = 1$$

From this we see that P_n must coincide both with y_1 and y_2 up to a factor. Namely, we have:

$$\begin{aligned} P_n(\zeta) &= F\left(-n, n+1, 1, \frac{1-\zeta}{2}\right) \\ &= (-1)^n F\left(-n, n+1, 1, \frac{1+\zeta}{2}\right), \end{aligned} \tag{24a}$$

which also yields the correct normalization $P_n(1) = 1$ for $\zeta = +1$. The series (24a) for P_n breaks off as does every hypergeometric series with negative integral α or β : since $\alpha = -n$ we have that P_n is a polynomial of degree n (the coefficients of $(1 \mp \zeta)^{n+1}$ and of all subsequent powers contain the factor $\alpha + n = -n + n = 0$ in the numerator). We remark that the series for P_n in terms of $1 - \zeta$ is simpler (since it is hypergeometric) than the series in terms of ζ. The latter reads

$$P_n = 1 + \frac{(-n)(n+1)}{1 \cdot 1} \frac{1-\zeta}{2}$$

$$(24\text{b}) \quad + \frac{(-n)(-n+1)(n+1)(n+2)}{2!\,2!}\left(\frac{1-\zeta}{2}\right)^2 + \cdots + (-1)^n \frac{(2n)!}{n!\,n!}\left(\frac{1-\zeta}{2}\right)^n$$

$$= \sum_{p=0}^{n} (-1)^p \frac{(n+p)!}{(n-p)!} \frac{1}{p!\,p!} \left(\frac{1-\zeta}{2}\right)^p .$$

The associated P_n^m can also be represented by a hypergeometric series. We merely have to consider the general relation which is obtained from (23) by termwise differentiation:

$$(25) \qquad \frac{d}{dz} F(\alpha, \beta, \gamma, z) = \frac{\alpha\beta}{\gamma} F(\alpha+1, \beta+1, \gamma+1, z).$$

Hence for positive m we obtain the representation

$$(26) \qquad P_n^m(\zeta) = C(1-\zeta^2)^{m/2} F\left(m-n,\ m+n+1,\ m+1,\ \frac{1-\zeta}{2}\right), \quad C = \frac{(n+m)!}{2^m\, m!\,(n-m)!}.$$

from (22.18). For the negative integral m this representation breaks down, but can be extended to that case by a limit process; the result then coincides with our general definition (22.18).

From the Gaussian hypergeometric function we derive the *confluent hypergeometric function*, which is of the utmost importance in wave mechanics. It depends only on two parameters α and γ since the third parameter β is subjected to the following limit process:

$$(27) \qquad \beta \to \infty, \ z \to 0, \ \beta z \to \varrho \quad (\varrho = \text{arbitrary finite number}).$$

We then obtain from (23)

$$(28) \qquad F(\alpha, \gamma, \varrho) = 1 + \frac{\alpha}{\gamma}\frac{\varrho}{1} + \frac{\alpha(\alpha+1)}{\gamma(\gamma+1)}\frac{\varrho^2}{2!} + \cdots$$

and considering the fact that

$$\frac{d}{dz} = \frac{d}{d\varrho} \cdot \frac{d\varrho}{dz} = \beta \frac{d}{d\varrho}$$

we obtain the corresponding differential equation from (20) after dividing by β :

$$\varrho \frac{d^2F}{d\varrho^2} + (\gamma - \varrho)\frac{dF}{d\varrho} - \alpha F = 0. \tag{29}$$

We shall encounter this equation again in connection with the eigenfunctions of hydrogen in wave mechanics.

E. Spherical Harmonics of Non-Integral Index

Our representation must still be completed in two directions. We have been restricted so far to the case of *integral numbers n and m* and to *functions* P_n^m, *which were finite throughout.* Both these restrictions were suggested by the connection with potential theory.

Concerning the first point, we see that for non-integral n the hypergeometric series at the points $\zeta = \pm 1$ in ascending powers of $1 \mp \zeta$ does not break off as it does in the case of integral n. The solution, which is regular at the north pole $\zeta = +1$, diverges at the south pole $\zeta = -1$ and vice versa. Thus (24a) is valid for integral n only. The "requirement of finiteness all over the sphere" can therefore be satisfied only for integral n. The possibility of non-integral m is excluded by the requirement of uniqueness with respect to the φ-coordinate.

The type of singularity of $P_n(\zeta)$ for non-integral n can be deduced from the general theory of hypergeometric series. We prefer, however, to deduce it by direct calculation.

According to their original definition in (22.3) the P_n are the coefficients of a Taylor series which progresses in powers of $t = r/r_0$; hence for integral n:

$$P_n(\zeta) = \frac{1}{n!}\frac{d^n}{dt^n}\frac{1}{\sqrt{1-2\zeta t+t^2}} \quad \text{at} \quad t = 0. \tag{30}$$

According to Cauchy's theorem this can be written as:

$$P_n(\zeta) = \frac{1}{2\pi i}\oint \frac{dt}{t^{n+1}}\frac{1}{\sqrt{1-2\zeta t+t^2}}. \tag{30a}$$

This representation also holds for non-integral n, except that due to the many-valued character of t^{-n-1} we have to perform a branch cut, e.g., from $t = -\infty$ to $t = 0$, and that the path of integration is now a loop which starts on the negative side of the cut at $t = -\infty$, then circles the point $t = 0$ in a counterclockwise direction and ends on the positive side of the cut at $t = -\infty$. It is clear that for this definition the differential equation of P_n is satisfied regardless of whether or not n is integral. Equation (30a) defines that particular solution of the differential equation which is regular at $\zeta = 1$ and satisfies the normalizing condition $P_n(1) = 1$. Namely for $\zeta = 1$ we obtain from (30a)

$$P_n(1) = -\frac{1}{2\pi i}\oint \frac{dt}{t^{n+1}}\frac{1}{t-1}. \tag{30b}$$

The integrand now has a simple pole at $t = 1$. The loop described above can now be deformed into a path which circles the pole in a clockwise direction. According to Cauchy's theorem the integral then has the value $-2\pi i$, and hence the right side of (30b) has the required value $+1$.

For $\vartheta > 0$ the integrand of (30a) has two further branch points that are due to the square root in the denominator and lie on the unit circle of the t-plane at

$$t = e^{i\vartheta} \quad \text{and} \quad t = e^{-i\vartheta}$$

We connect these branch points by a branch cut, e.g., along the unit circle. The path of integration may not cross this cut either. For $\vartheta = \pi - \delta$, $\delta \ll 1$, the endpoints of the cut approach the negative real axis and restrict the path of integration between them. This explains why (30a) becomes singular for $\delta \to 0$, or in other words for $\vartheta \to \pi$, $\zeta \to -1$.

In order to discuss this singularity we write

$$t = e^{i\pi}(1+\tau) \quad \text{and} \quad t = e^{-i\pi}(1+\tau);$$

in the neighborhood of the point $t = -1$ on the upper and lower edge of our branch cut respectively. Then, except for terms of higher order in τ and δ, the square root in (30a) becomes

$$\sqrt{1 - 2\zeta t + t^2} = \sqrt{\tau^2 + \delta^2}.$$

Hence for small δ only the neighborhood of $\tau = 0$ contributes to our limiting value as $\vartheta \to \pi$. We may, therefore, restrict the integration over the upper and lower edges to the small region between

$$\tau = +\varepsilon \quad \text{and} \quad \tau = -\varepsilon$$

and considering the orientation on the two edges we may write:

$$\frac{dt}{t^{n+1}} = \begin{cases} e^{+i\pi n}\, d\tau & \text{lower edge} \\ -e^{-i\pi n}\, d\tau & \text{upper edge,} \end{cases}$$

hence:

$$(31)\qquad P_n(\zeta) = \frac{e^{+i\pi n} - e^{-i\pi n}}{2\pi i} \int\limits_{+\varepsilon}^{-\varepsilon} \frac{d\tau}{\sqrt{\tau^2+\delta^2}} = \frac{\sin n\pi}{\pi} \int\limits_{+\varepsilon}^{-\varepsilon} \frac{d\tau}{\sqrt{\tau^2+\delta^2}}.$$

Now according to a well known formula we have

$$\int \frac{d\tau}{\sqrt{\tau^2+\delta^2}} = \log(\tau + \sqrt{\tau^2+\delta^2})$$

for undetermined upper and lower limits of integration. Hence for the definite integral in (31) we have

$$\log(-\varepsilon + \sqrt{\varepsilon^2+\delta^2}) - \log(+\varepsilon + \sqrt{\varepsilon^2+\delta^2})$$

and for $\delta \ll \varepsilon$

$$\log \frac{1}{2}\frac{\delta^2}{\varepsilon} - \log\left(2\varepsilon + \frac{1}{2}\frac{\delta^2}{\varepsilon}\right).$$

In the limit $\delta \to 0$ we have $\log \delta^2$ as the leading term; hence we obtain from (31)

$$(32)\qquad \lim_{\zeta \to -1} P_n(\zeta) = \frac{\sin n\pi}{\pi} \log \delta^2 + \cdots,$$

The terms[22] . . . which have been omitted here reduce for $\delta \to 0$ to a finite constant which is of course independent of ε

F. Spherical Harmonics of the Second Kind

At the beginning of Section E we saw that for non-integral n two different solutions P_n of the hypergeometric differential equation exist. Only for integral n do these solutions coincide. But in this latter case, too, a second solution must exist in addition to the everywhere regular solution found above. This solution will be singular at the points $\zeta = \pm 1$ We call it a *spherical harmonic of the second kind* and denote it by Q_n.

The type of singularity can be determined from general theorems. In (21a) we saw that the quadratic equation for the exponent λ for the case of spherical harmonics $(\gamma = 1)$ has the *double root* $\lambda = 0$. By a passage to the limit we see that this indicates a logarithmic singularity for $\zeta = \pm 1$. Just as in the case of spherical harmonics of the first kind, we obtain detailed information about the *spherical harmonics*

[22] They are computed in Hobson's textbook, equation (53). p. 225, which was quoted on p. 129 above.

of the second kind Q_n from a *generating function*[23] (C. Neumann):

$$\frac{1}{\eta-\zeta}=\sum_{n=0}^{\infty}\left(n+\frac{1}{2}\right)Q_n(\eta)\,P_n(\zeta)\,. \tag{33}$$

Hence the $Q_n(\eta)$ are defined as coefficients in the expansion of $1/(\eta-\zeta)$ in the P_n, and therefore they are given by the following *integral representation* (F. Neumann):

$$Q_n(\eta)=A.M.\int_{-1}^{+1}P_n(\zeta)\frac{d\zeta}{\eta-\zeta}\,. \tag{34}$$

In this formula the path of integration is to avoid the singular point $\zeta=\eta$ by going around it through the complex domain both to the right and to the left, and the symbol $A.M.$, which we shall omit in the future, indicates that we have to take the arithmetic mean of the two values obtained (this is identical with the so-called "principal value" of the integral). The fact that this avoidance of the singularity is not possible at the limits $\zeta=\pm 1$ implies the above-mentioned logarithmic singularity.

That $Q_n(\eta)$ satisfies the differential equation (24) (written in terms of η), is to be expected from the symmetry of the defining equation in η, ζ and Q,P, but it can also be demonstrated directly as follows: we abbreviate equation (24) to

$$L_\zeta\{P\}=0;\qquad L_\zeta=\frac{d}{d\zeta}(1-\zeta^2)\frac{d}{d\zeta}+n(n+1),$$

and by $L_\eta\{Q\}$ we mean the analogous expression written in terms of η and Q; we further note the identity

$$L_\eta\left\{\frac{1}{\eta-\zeta}\right\}=L_\zeta\left\{\frac{1}{\eta-\zeta}\right\}.$$

Then from (34) we have

$$L_\eta\{Q_n\}=\int_{-1}^{+1}P_n(\zeta)\,L_\zeta\left\{\frac{1}{\eta-\zeta}\right\}d\zeta \tag{35}$$

We integrate by parts twice, then the terms which are due to the limits $\zeta=\pm 1$ vanish on account of the factor $1-\zeta^2$ in L_ζ and we obtain

$$L_\eta\{Q_n(\eta)\}=\int_{-1}^{+1}L_\zeta\{P_n(\zeta)\}\frac{d\zeta}{\eta-\zeta}=0\ \text{, q.e.d.} \tag{36}$$

[23] Usually double this function is used, so that in (33) we have $2n+1$ instead of $n+½$. Correspondingly our $Q_n(\eta)$ differ from the customary ones by the factor 2.

It is now easy to compute the first $Q_n(\eta)$ in terms of the known $P_n(\eta)$ with the help of (34. We deduce for $|\eta| < 1$

$$\text{from } P_0 = 1: \qquad Q_0 = \log\frac{1+\eta}{1-\eta},$$

$$\text{from } P_1 = \zeta: \qquad Q_1 = -2 + \eta\log\frac{1+\eta}{1-\eta}.$$

The general law is (Christoffel):

$$Q_n(\eta) = \Pi + P_n(\eta)\log\frac{1+\eta}{1-\eta}, \tag{37}$$

where Π is a polynomial of degree $n-1$ which is composed additively from all those P_{n-2k-1} for which the index is non-negative. Finally we obtain from (34), through m-fold differentiation with respect to η and multiplication by $(1-\eta^2)^{m/2}$ (which is analogous to $\sin^m\vartheta$),

$$Q_n^m(\eta) = (-1)^m m!\,(1-\eta^2)^{m/2}\int_{-1}^{+1}\frac{P_n(\zeta)}{(\eta-\zeta)^{m+1}}\,d\zeta. \tag{38}$$

Appendix I

Reflection on a Circular-Cylindrical or Spherical Mirror

Referring back to Fig. 8 and the notations defined there we continue the treatment of the problem which we started in §6.

a) Circular-cylindrical metal mirror. The incoming wave (electric vector which is perpendicular to the plane of the drawing) is

$$w = e^{ikr\cos\varphi} \approx I_0(kr) + 2\sum_{n=1}^{N} i^n I_n(kr)\cos n\varphi. \tag{1}$$

(see (21.2b)). This representation holds for the entire r,φ-plane and for $r = a$ it defines the function $-f(\varphi)$ in (6.4). The sum of $N+1$ terms on the right is the best approximation to w that can be obtained by the method of least squares; the fact that the coefficients in this sum are the same as those in the exact non-truncated series (21.2b) follows from the "finality" of the Fourier series. We write the radiation which is reflected (diffracted, scattered) by the mirror as the sum of $N+1$ particular solutions of the differential equation

$$\Delta u + k^2 u = 0$$

for $r < a$ in the form:

$$u = \sum_{0}^{N} C_n \frac{I_n(kr)}{I_n(ka)}\cos n\varphi; \tag{2}$$

(since the solution must be continuous for $r = 0$ only the I_n can occur in the representation; the sine terms disappear on account of the symmetry of the incoming wave with respect to $\varphi = 0$). The denominator $I_n(ka)$ is used for the sake of convenience and merely influences the meaning of the constants C_n which are as yet undetermined. The same holds for the denominators in equation (3) below.

We write the radiation which is scattered by the mirror to the outside $r > a$ as the following sum of $N + 1$ particular solutions of the wave equation:

$$(3) \qquad v = \sum_{0}^{N} D_n \frac{H_n^1(kr)}{H_n^1(ka)} \cos n\varphi .$$

The time dependence of the whole process should be thought of as given by $\exp(-i\omega t)$; hence only the H^1 occur; the H^2 would correspond to absorbed waves. As we saw in §6 the boundary conditions (6.8) to (6.11) imply $C_n = D_n$ and, according to the method of least squares, the system of linear equations (6.12). The constant γ_n which occurs there is determined from (6.7), (6.11a) and the equations (3), (4) above as

$$(4) \qquad \gamma_n = ka\left(\frac{I_n'(ka)}{I_n(ka)} - \frac{H_n^{1\prime}(ka)}{H_n^1(ka)}\right).$$

According to a well known theorem in the theory of linear differential equations we can rewrite this in the simpler form (see exercise IV.8)

$$(5) \qquad \gamma_n = \frac{-2i/\pi}{I_n(ka)\, H_n^1(ka)} .$$

We introduce the notation:

$$(6) \qquad a_{nm} = \int_{\alpha}^{\pi} \cos n\varphi \cos m\varphi \, d\varphi$$

and obtain by a simple transformation

$$(7) \qquad \int_{0}^{\alpha} \cos n\varphi \cos m\varphi \, d\varphi = \frac{\pi}{(2)} \delta_{nm} - a_{nm} .$$

Here and in the following the symbol (2) stands for the number 2 when $n > 0$ and for the number 1 when $n = 0$. Then the left side of (6.12) becomes

$$(8) \qquad \frac{\pi}{(2)} C_m + \sum_{n=0}^{N} a_{nm} (\gamma_n \gamma_m - 1) C_n$$

Setting $f(\varphi) = -w$, where w is as in equation (1), and $r = a$, we obtain for the right side of (6.12):

(8a)
$$-\pi i^m I_m(ka) + \sum_{n=0}^{N} a_{nm}(2)\, i^n I_n(ka).$$

Hence the system (6.12) becomes

(9)
$$C_m + (2)\, i^m I_m(ka) - \frac{(2)}{\pi}\sum_{n=0}^{N'} a_{nm}\{C_n + (2)\, i^n I_n(ka) - \gamma_n\gamma_m C_n\} = 0,$$

which must be satisfied for all $m = 0,1, \ldots, N$.

In order to discuss this system we first set $\alpha = \pi$, in other words we consider a complete circle (spatially speaking a closed, totally conductive, cylinder). According to (6) we then have $a_{nm} = 0$ and (9) yields

(10)
$$C_m = -(2)\, i^m I_m(ka).$$

This result is somewhat trivial. For, by substituting the value (10) of C_n for D_n in (3), we obtain the rigorous solution v of the corresponding scattering problem for $r > a$:

(11)
$$v = -\sum_{n=0}^{N} (2)\, i^n I_n(ka)\frac{H_n^1(kr)}{H_n^1(ka)}\cos n\varphi,$$

a radiated wave which, on the cylinder $r = a$, exactly cancels the incoming wave w of equation (1) and hence for $N \to \infty$ yields the rigorous solution of the scattering problem. In the same manner we obtain for u:

(12)
$$u = -\sum_{n=0}^{N} (2)\, i^n I_n(kr)\cos\varphi = -w.$$

This result is also trivial since in the interior of the closed circle $r = a$ we must have $u + v = 0$.

We now wish to investigate whether our equation (10) yields a useful approximation in the case $\alpha < \pi$, too. To this end we set

(13)
$$C_n = -(2)\, i^n I_n(ka) + \beta_n$$

where β_n is a correction term, and obtain from (9):

(14)
$$\beta_m - \frac{(2)}{\pi}\sum_{n=0}^{N} a_{nm}\{\beta_n(1-\gamma_n\gamma_m) + \gamma_n\gamma_m(2)\, i^n I_n(ka)\} = 0.$$

As in (6.12) this is a system of $N + 1$ (and in the limit infinitely many) linear equations with which we can do practically nothing. However if we assume that $\pi - \alpha$ is small, that is that the cylinder has only a *narrow slit*, then a_{nm} becomes small and the product $\beta_n a_{nm}$ becomes *small of the second order*. If we neglect this term then (14) becomes simply

(15)
$$\frac{\beta_m}{\gamma_m} = -\frac{(2)}{\pi}\sum_{n=0}^{N} (2)\, i^n a_{nm}\gamma_n I_n(ka),$$

which is an explicit value for β_m and from (13) yields an explicit value for C_m.

The narrowness of the slit necessary for this consideration can be estimated by a physical consideration: its width must be *small compared to the wavelength* of the incoming radiation; only in this case does the interior field go over continuously into the zero field of the closed cylinder. Hence we must have

$$\frac{(\pi-\alpha)a}{\lambda} < 1. \tag{16}$$

This condition can be satisfied only approximately for Hertz waves. In the properly optical case this approximation breaks down. This is the reason we spoke of a "quasi-optical case" on p. 29. In the well known Hertz experiment with concave mirrors, for which we have, say, $\alpha = \pi/2$, $\lambda = 200$ cm., $a = 50$ cm., equation (16) is approximately satisfied so that our system of approximation is justified.

b) *The sphere segment as an acoustic reflector.* In order to avoid discussions on vectors we deal with scalar acoustic waves instead of directed optical radiation. By w,u,v we mean the *velocity potentials* of the primary and secondary (reflected) radiation in the interior ($r < a$) and exterior ($r > a$) of the sphere. Let the sphere segment be given by $r = a$, $0 < \vartheta < \alpha$. According to (24.7) we write w in the form

$$w = \sum_{n=0}^{N} (2n+1)\, i^n \psi_n(kr)\, P_n(\cos\vartheta); \tag{17}$$

this is the best possible approximation of a plane wave $\exp(ikr\cos\vartheta)$ by $N+1$ spherical harmonics according to the method of least squares. We further write

$$u = \sum C_n \frac{\psi_n(kr)}{\psi_n'(ka)} P_n(\cos\vartheta) \qquad r < a, \tag{18}$$

$$v = \sum D_n \frac{\zeta_n(kr)}{\zeta_n'(ka)} P_n(\cos\vartheta) \qquad r > a, \tag{19}$$

where the constants C_n, D_n are as yet undetermined. Concerning the denominators see the remark to equation (2). The function ζ_n is the Bessel function with half-integral index which was defined in (21.15) and corresponds to the Hankel function H_n^1. On the sphere segment (which was assumed rigid) we have

$$\frac{\partial}{\partial n}(u+w) = \frac{\partial}{\partial n}(v+w) = 0 \quad \text{for} \quad 0 \leq \vartheta < \alpha \quad \text{and } r = a,$$

and for reasons of continuity we must have

$$u = v, \ \frac{\partial u}{\partial r} = \frac{\partial v}{\partial r} \ \text{ for } \ \alpha < \vartheta \leq \pi \ \text{ and } \ r = a.$$

This condition again leads to $D_n = C_n$ and to the equations:

$$\begin{aligned} \sum C_n P_n(\cos\vartheta) &= f(\vartheta), \qquad 0<\vartheta<\alpha, \\ \sum D_n \gamma_n P_n(\cos\vartheta) &= 0, \qquad \alpha<\vartheta<\pi. \end{aligned} \tag{20}$$

which are analogous to (6.10) and (6.11). Here $f(\vartheta)$ stands for the value of $-\partial w/\partial(kr)$ for $r = a$,

$$f(\vartheta) = -\left(\frac{\partial w}{\partial(kr)}\right)_{r=a}, \qquad w = e^{ikr\cos\vartheta} \tag{20 a}$$

and in analogy to (5)

$$\gamma_n = \frac{\psi_n(ka)}{\psi_n'(ka)} - \frac{\zeta_n(ka)}{\zeta_n'(ka)}. \tag{20 b}$$

Introducing the abbreviation

$$a_{nm} = \int_\alpha^\pi P_n P_m \sin\vartheta\, d\vartheta$$

we obtain

$$\int_0^\alpha P_n P_m \sin\vartheta\, d\vartheta = \frac{\delta_{nm}}{m+\frac{1}{2}} - a_{nm} \tag{21 a}$$

and from (20a)

$$\begin{aligned} \int_0^\alpha f(\vartheta)\, P_m(\cos\vartheta)\sin\vartheta\, d\vartheta &= -\sum_{n=0}^N (2n+1)\, i^n \psi_n'(ka) \int_0^\alpha P_n P_m \sin\vartheta\, d\vartheta \\ &= -2\, i^m \psi_m'(ka) + \sum_{n=0}^N a_{nm}(2n+1)\, i^n \psi_n'(ka). \end{aligned} \tag{21 b}$$

In analogy to (6.12) the method of least squares now yields the following system of equations for the C_n:

$$\sum_{n=0}^N C_n \left\{ \int_0^\alpha P_n P_m \sin\vartheta\, d\vartheta + \gamma_n\gamma_m \int_\alpha^\pi P_n P_m \sin\vartheta\, d\vartheta \right\} = \int_0^\alpha f(\vartheta)\, P_m \sin\vartheta\, d\vartheta, \tag{22}$$

which holds for $m = 0,1, \ldots, N$; due to (21a,b) this becomes

$$\begin{aligned} \frac{C_m}{m+\frac{1}{2}} - \sum_{n=0}^N a_{nm} C_n (1-\gamma_n\gamma_m) \\ = -2\, i^m \psi_m'(ka) + \sum_{n=0}^N a_{nm}(2n+1)\, i^n \psi_n'(ka). \end{aligned} \tag{22a}$$

which, as in equation (9), can be rearranged to

$$\begin{aligned} \frac{1}{m+\frac{1}{2}} [C_m + (2m+1)\, i^m \psi_m'(ka)] \\ - \sum_{n=0}^N a_{nm} \{C_n + (2n+1)\, i^n \psi_n'(ka) - \gamma_n\gamma_m C_n\} = 0. \end{aligned} \tag{23}$$

We again start with the limiting case $\alpha = \pi$ of a closed sphere in which $a_{nm} = 0$. Then (23) yields

(24) $$C_m = -(2m+1)\, i^m \psi_m'(ka).$$

Substituting this value in (19) we obtain the rigorous solution v of our acoustic problem for $r > a$, namely, the reflection of an incoming wave on a closed sphere. In the interior $r < a$ of the sphere we obtain, by substituting C_m in (18), a field u which, as it should be, is the negative of the field w of the incoming wave.

The next problem is that of a spherical surface with a circular hole in the neighborhood of $\vartheta = \pi$. By setting

(25) $$C_m = -(2m+1)\, i^m \psi_m'(ka) + \beta_m,$$

and ignoring the product term of second order $a_{nm}\beta_n$ we obtain from (23)

(26) $$\frac{1}{m+\frac{1}{2}} \frac{\beta_m}{\gamma_m} = \sum_{n=0}^{N} (2n+1)\, i^n a_{nm} \gamma_n \psi_n'(ka),$$

which is an explicit computation of the correction term β_m and hence of the coefficients C_m. The reader should compare this result with the analogous result for the problem of the cylinder in equation (15). Just as the width of the slit there, so the diameter of the circular hole here must be small compared to the wavelength of the incoming radiation. Hence, here too, we can treat only a "quasi-acoustical" problem, the problem of *infra-sound*, which is very far from the more interesting problem of *ultra-sound*.

Our aim in this somewhat sketchy appendix has been to show that the method of least squares may be applied successfully even in some cases in which our condition of finality for the computation of the coefficients C_m is not satisfied.

Appendix II

Additions to the Riemann Problem of Sound Waves in §11

The purpose of this appendix is to fill a gap which we left in §11; namely we shall prove that the expression (11.10)

(1) $$v = \left(\frac{\xi+\eta}{x+y}\right)^a F(a+1, -a, 1, z), \qquad z = -\frac{(x-\xi)(y-\eta)}{(x+y)(\xi+\eta)},$$

where F stands for the hypergeometric series, satisfies the differential equation

(2) $$M(v) = \frac{\partial^2 v}{\partial x\, \partial y} + \frac{a}{x+y}\left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y}\right) - \frac{2av}{(x+y)^2} = 0$$

which is derived from (11.2) and (11.8). Riemann was able to prove this by his general transformation theory for hypergeometric functions. However we shall proceed in an elementary fashion, by considering the function F in (1) as an unknown function and then by substituting (1) in (2) and deducing a differential equation for $F(z)$. By proving the latter identical with the differential equation (24.20) of the hypergeometric function we verify equation (1) and the determination of the parameters of the hypergeometric function which are contained in it.

First we deduce from (1)

$$\frac{\partial v}{\partial x} = \left(\frac{\xi+\eta}{x+y}\right)^a \left(\frac{-a}{x+y} F(z) + \frac{\partial z}{\partial x} F'(z)\right),$$
$$\frac{\partial v}{\partial y} = \left(\frac{\xi+\eta}{x+y}\right)^a \left(\frac{-a}{x+y} F(z) + \frac{\partial z}{\partial y} F'(z)\right)$$

and hence we obtain as the sum of the last two terms in (2)

$$\frac{a}{x+y}\left(\frac{\xi+\eta}{x+y}\right)^a \left[-\frac{2a+2}{x+y} F(z) + \left(\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\right) F'(z)\right]. \tag{3}$$

As the first term of (2) we then obtain

$$\frac{a(a+1)(\xi+\eta)^a}{(x+y)^{a+2}} F(z) - \frac{a(\xi+\eta)^a}{(x+y)^{a+1}}\left(\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\right) F'(z) + \left(\frac{\xi+\eta}{x+y}\right)^a \left(\frac{\partial^2 z}{\partial x\,\partial y} F'(z) + \frac{\partial z}{\partial x}\frac{\partial z}{\partial y} F''(z)\right). \tag{4}$$

The first two terms of (4) combine and cancel respectively with the two terms of (3). Hence (2) becomes

$$\frac{\partial z}{\partial x}\frac{\partial z}{\partial y} F''(z) + \frac{\partial^2 z}{\partial x\,\partial y} F'(z) - \frac{a(a+1)}{(x+y)^2} F = 0. \tag{5}$$

The derivatives of z here can be expressed as follows:

$$\frac{\partial z}{\partial x}\frac{\partial z}{\partial y} = \frac{1}{(x+y)^2}(z^2 - z), \tag{6}$$

$$\frac{\partial^2 z}{\partial x\,\partial y} = \frac{2z-1}{(x+y)^2}. \tag{7}$$

Due to (6) and (7) equation (5) becomes

$$z(1-z)F'' + (1-2z)F' + a(a+1)F = 0.$$

This is indeed the same as equation (24.20) if we substitute

$$\alpha = a+1, \quad \beta = -a, \quad \gamma = 1.$$

as in (1). This completes the discussion of the problem of §11.

CHAPTER V

Eigenfunctions and Eigen Values

In this chapter we shall develop Fourier's methods to their greatest generality and thereby open up the boundary value problems of physics to mathematical treatment. The most striking demonstration of the power of these methods was given in 1926 when Erwin Schrödinger recognized the quantum numbers as eigen values of his wave equation and thereby put the tools of modern analysis at the service of atomic physics. It was fortunate that he had the aid of his Zürich colleague Hermann Weyl who had, as the greatest pupil and later the successor of Hilbert in Göttingen, an essential part in the development of the theory of integral equations. However we should note that, while the viewpoint of *integral equations* is important for the rigorous mathematical foundation, in particular for the existence proofs for the eigenfunctions and their eigen values, the older viewpoint of *partial differential equations* leads to the same concepts in a natural manner. We shall start by demonstrating this with an example which was known long before integral equations.

§ 25. Eigen Values and Eigenfunctions of the Vibrating Membrane

The subject of the following consideration is a membrane without proper elasticity (see p. 33) which is clamped into a frame whose resistance to distortion is entirely due to the stresses working on its edge. We consider these stresses as perpendicular to the edge in the plane of the membrane. For the deformed membrane this results in a pressure N which acts perpendicular to the surface and is equal to T times the mean curvature of the membrane, and hence is equal to $T\,\Delta u$ for a small deformation u. The wave equation (7.4) for a pure harmonic oscillation of frequency ω then yields

$$-\sigma\,\omega^2\,u = T\,\Delta u, \qquad \sigma = \text{surface density.}$$

This we rewrite in the customary form

$$\Delta u + k^2\,u = 0, \qquad k^2 = \frac{\sigma\,\omega^2}{T}. \tag{1}$$

If we do not consider k^2 as constant but as an arbitrary function $F(x,y)$, then according to (10.6) this is the general linear *self-adjoint* elliptic differential equation of second order in two variables in its normal form.

The non-trivial solutions of (1) which satisfy the boundary condition $u = 0$ are called *eigenfunctions* and the corresponding k are called the *eigen values* of the problem. If k^2 or $F(x,y)$ were negative then no eigen values would exist, as we saw in the introduction to exercise II.2. The fact that eigen values do exist for positive k^2, namely, an *infinite number*, can be shown first for the simplest examples.

a) The rectangle $0 \leq x \leq a,\ 0 \leq y \leq b$. The boundary conditions are satisfied by

$$u = u_{nm} = \sin n\pi \frac{x}{a} \sin m\pi \frac{y}{b}, \begin{cases} n = 1, 2, \cdots \infty, \\ m = 1, 2, \cdots \infty. \end{cases} \tag{2}$$

From the differential equation we then have

$$k = k_{nm} = \pi \sqrt{\frac{n^2}{a^2} + \frac{m^2}{b^2}}. \tag{2 a}$$

We shall ignore the constant factor by which the solutions can be multiplied. We first assume a and b to be *incommensurable*. Then all the k_{nm} are different and only *one* eigenfunction u corresponds to each k. The number of eigen values is *infinite*.

b) Circle, circular ring, circular sector. For the *full circle* $0 \leq r \leq a$ we can write:

$$u = I_m(kr)\, e^{\pm im\varphi}, \quad m = 0, 1, 2, \ldots \infty, \tag{3}$$

where k satisfies the boundary condition

$$I_m(ka) = 0. \tag{3 a}$$

Since this equation has infinitely many roots (see Fig. 21) there are again *infinitely many eigen values* $k = k_{nm}$. The roots of (3a) are all different, but for $m > 0$ there are two eigenfunctions for each eigen value corresponding to the different signs in (3), or, in other words, corresponding to the double possibility $\begin{smallmatrix}\cos \\ \sin\end{smallmatrix}\, m\varphi$. We say that the problem is *degenerate* for $m > 0$; in our case it is *simply degenerate*. According to (20.4b) the (non-degenerate) basic tone of the circular membrane is $k_{10} = 2.40/a$.

For the *circular ring* $b \leq r \leq a$ we write

$$u = [I_m(kr) + c\, N_m(kr)]\, e^{\pm im\varphi}. \tag{4}$$

Here we need both particular solutions I and N of the Bessel differential equation (we could, of course, consider H^1 and H^2 instead) in order to be able to satisfy the two boundary conditions:

$$(4a)\qquad \begin{aligned} I_m(k\,a) + c\,N_m(k\,a) &= 0,\\ I_m(k\,b) + c\,N_m(k\,b) &= 0. \end{aligned}$$

Here too there exists an infinite number of different k_{nm} with their associated c_{nm}. This problem, too, is simply degenerate when $m > 0$, since, according to (4), there are then two different u_{nm} for each k_{nm}.

For the *circular sector* $0 \leq r \leq a,\ 0 \leq \varphi \leq \alpha$ we set

$$(5)\qquad u = I_\mu(k\,r)\,\sin\mu\varphi, \qquad \mu = m\frac{\pi}{\alpha},$$

where the k are determined by the condition $I_\mu(ka) = 0$. Infinitely many eigen values $k = k_{nm}$ exist; the problem is not degenerate.

The most general region which can be treated in this manner is the *circular ring sector* $b \leq r \leq a,\ 0 \leq \varphi \leq \alpha$, which is bounded by two circular arcs and two radii.

c) Ellipse and elliptic-hyperbolic curvilinear quadrangle. The wave equation (1) written in elliptic coordinates ξ, η can be separated (see v.II exercise IV.3) and leads to a so-called Mathieu equation in each coordinate. The solution $\xi = \text{const}$ yields the ellipses which belong to the family of curves; $\eta = \text{const}$ yields the hyperbolas of the family. For the full ellipse we have, in addition to the boundary condition $u = 0$, a condition of continuity for $\xi = 0$ (focal line) and the condition of periodicity for $\eta = \pm\pi$. The determination of the eigen values leads to complicated transcendental equations which we cannot discuss here. The most general region of this kind is the curvilinear quadrangle whose boundary consists of two elliptic arcs and two arcs of hyperbolas which are confocal with the former.

The simple examples which we considered here are special cases of the *fundamental theorem* of the theory of oscillating systems with infinitely many degrees of freedom and their eigenfunctions: *For an arbitrary region an infinite sequence of eigen values k exists for which there is a solution of the corresponding differential equation* $\Delta u + k^2 u = 0$ *which is continuous in the interior of the region and satisfies the boundary condition* $u = 0$ (or any of the other boundary conditions on p. 63).[1] The problem of finding a rigorous proof for this theorem has repeatedly challenged the ingenuity of mathematicians, starting with Poincaré's great work (*Rendic. Circ. Math. di Palermo*, 1894) and culminating in the

[1] The same theorem holds for the eigen value λ of the general self-adjoint differential equation $\Delta u + \lambda F(x, y)\,u = 0,\ F > 0$.

Fredholm-Hilbert theory of integral equations. Here we must be satisfied with proving the related theorem for mechanical systems with a finite number of degrees of freedom: *A system with f degrees of freedom which is in stable equilibrium, can have exactly f linearly independent small (or more precisely, infinitely small) sine-like oscillations about this state.*

We write the kinetic energy for the neighborhood of a state of equilibrium $q_1 = q_2 = \ldots = q_f = 0$ in the form:

$$T = \frac{1}{2} \sum \sum a_{nm} \dot{q}_n \dot{q}_m .$$

Because q is so small we consider the a_{nm} as constants. At the same time the potential energy V becomes a quadratic form in the q_n with constant coefficients since the linear terms $\partial V/\partial q_n$ vanish in the expansion of V in terms of the q_i around the state of equilibrium

$$V - V_0 = \frac{1}{2} \sum \sum b_{nm} q_n q_m .$$

Now it is always possible to transform *both* the above quadratic forms simultaneously into sums of squares by a linear transformation (transformation to principal axes of quadratic surfaces). Performing this transformation we obtain:

$$T = \frac{1}{2} \sum a_n \dot{x}_n^2 , \qquad V - V_0 = \frac{1}{2} \sum b_n x_n^2 .$$

The new coordinates x_n are called *normal coordinates* of the system. According to the Lagrange equation we then have

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{x}_n} = -\frac{\partial V}{\partial x_n}, \quad \text{hence} \quad a_n \ddot{x}_n = -b_n x_n .$$

T is a positive definite quadratic form and so is $V - V_0$ for a stable equilibrium; hence the a_n and b_n are positive. Thus for every normal coordinate we obtain a stable oscillation

$$x_n = c_n e^{i\omega_n t} \quad \text{with} \quad \omega_n^2 = \frac{b_n}{a_n} > 0 ,$$

which gives as many oscillations as there are degrees of freedom. In the limit $f \to \infty$ there corresponds an eigen value k_n to every ω_n, and to the totality of $q_1, \ldots, q_n$ that belong to the individual x_n there now corresponds the eigenfunction u_f. The k and the ω_n are both *real.*

We point out that the fact that the k are real can also be proved directly from the differential equation without passing to the limit. If a k were complex then the corresponding u would be complex and the conjugate function u^* would have to satisfy the conjugate differential equation $\Delta u^* + k^{*2} u^* = 0$ with the boundary condition $u^* = 0$. From Green's theorem

(6) $$\int (u\,\Delta u^* - u^*\,\Delta u)\,d\sigma = \int \left(u\frac{\partial u^*}{\partial n} - u^*\frac{\partial u}{\partial n}\right) ds,$$

where the right side vanishes due to the boundary conditions; it follows that

$$(k^2 - k^{*2}) \int u\,u^*\,d\sigma = 0.$$

But uu^* is always $\geqq 0$; hence the integral cannot vanish and hence we must have $k = k^*$, and k must be real. The physical meaning of the real character of the k is that under our conditions the oscillation process is always free from absorption.

Up to now we have assumed our problem to be *non-degenerate.* However, for the perturbation theory of wave mechanics the degenerate cases are of special interest. We return to our example of the rectangle and no longer assume the sides a,b, to be incommensurable. This is certainly the case for the *square* $a = b$. Then we obtain from (2a)

$$k_{nm} = \frac{\pi}{a}\sqrt{n^2 + m^2}, \quad \text{hence} \quad k_{nm} = k_{mn};$$

but according to (2) we have $u_{nm} \neq u_{mn}$, unless $n = m$, namely,

$$u_{nm} = \sin n\pi\frac{x}{a}\,\sin m\pi\frac{y}{a},$$

but

$$u_{mn} = \sin m\pi\frac{x}{a}\,\sin n\pi\frac{y}{a},$$

All oscillations with $n \neq m$ are therefore (at least) *simply degenerate,* since two different types of oscillations u_{nm} and u_{mn} correspond to the same k_{mn}. Only the basic oscillation k_{11} and its overtones $k_{nn} = n\,k_{11}$ (which in this special case are harmonic) are *non-degenerate.*

Let us examine somewhat more closely the cases $n = 1$, $m = 2$ and $n = 2, m = 1$ (hence $k_{12} = k_{21} = \sqrt{5}\,\pi/a$). In Figs. 23 and 24 we characterize the corresponding eigenfunctions by their nodal lines. These are the lines $u = 0$ in which powder strewn on the membrane would collect. Together with u_{12} and u_{21} we have, belonging to $k_{12} = k_{21}$, the eigenfunctions

(7) $$u = u_{12} + \lambda\,u_{21},$$

where λ is an arbitrary constant. By a continuous

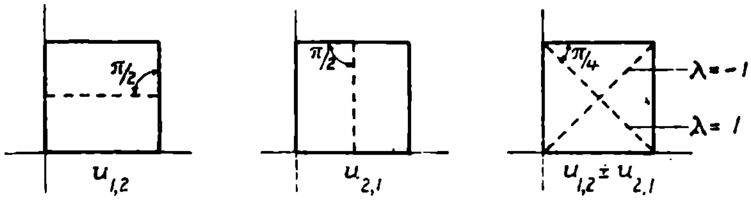


Fig. 23. Simple degeneration in the case of a quadratic membrane for $n = 1$. $m = 2$ or $m = 1$, $n = 2$. The diagonals are the nodal lines for $\lambda = \pm 1$.

deformation of λ the form of the nodal lines within the family (7) is continuously deformed. We compute the linear combinations with $\lambda = \pm 1$:

$$u = \sin \pi \frac{x}{a} \sin 2\pi \frac{y}{a} \pm \sin 2\pi \frac{x}{a} \sin \pi \frac{y}{a}$$

$$= 2 \sin \pi \frac{x}{a} \sin \pi \frac{y}{a}\left(\cos \pi \frac{y}{a} \pm \cos \pi \frac{x}{a}\right).$$

From the last expression here we see that the diagonal $y = x$ is a nodal line of $\lambda = -1$ while the other diagonal $y = a - x$ belongs to $\lambda = +1$. Fig. 24 shows the behavior of the lines for arbitrary values of the parameter λ.

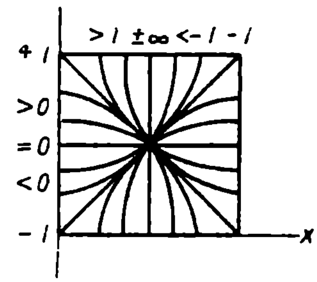


Fig. 24. Total picture of the possible nodal lines for the quadratic membrane. The numbers on the left and on top are the values of the parameter λ in (7).

Under certain conditions higher degenerations occur in the case of the quadratic membrane. For example, if we have

$$n_1^2 + m_1^2 = n_2^2 + m_2^2 ;$$

then for the eigen value

$$k = \frac{\pi}{a}\sqrt{n_1^2 + m_1^2} = \frac{\pi}{a}\sqrt{n_2^2 + m_2^2}$$

we have four linearly independent eigenfunctions

$$u_{n_1 m_1}, \quad u_{n_2 m_2}, \quad u_{m_1 n_1}, \quad u_{m_2 n_2}.$$

Hence we have a case of triple degeneration. The higher degeneration here depends on whether or not a number can be expressed as the sum of two squares in more than one way, as for example

$$65 = 1^2 + 8^2 = 4^2 + 7^2.$$

According to Gauss' *Disquisitiones Arithmeticae* this is the case for every sum of two squares among whose prime factors there are at least two different ones of the form $4n + 1$. Such primes permit the complex decomposition

$$4n + 1 = (a + bi)(a - bi)$$

with integral a,b; and the different groupings of the complex factors lead to different representations as a sum of squares. In our example $65 = 5.13$ we have

$$5 = (1 + 2i)(1 - 2i) \text{ and } 13 = (2 + 3i)(2 - 3i)$$

and hence

$$65 = \begin{cases} (1+2i)\ (2+3i)\cdot(1-2i)\ (2-3i) = (-4+7i)\ (-4-7i) \\ \qquad = 4^2 + 7^2, \\ (1+2i)\ (2-3i)\cdot(1-2i)\ (2+3i) = (8+i)\cdot(8-i) \\ \qquad = 8^2 + 1^2. \end{cases}$$

For any two eigenfunctions u,u' with $k \neq k'$ we have the *orthogonality theorem*

$$\int u\,u'\,d\sigma = 0 \tag{8}$$

as a result of Green's theorem. The proof is the same as in (6) if we replace u^* by u'. But this deduction fails if u and u' belong to the same degenerate state, in other words if $k = k'$.

In order to avoid cumbersome considerations of special cases, it is desirable to force orthogonality also in the degenerate cases. It will prove convenient to introduce the abbreviation of Courant-Hilbert[2] for the integral in (8):

$$\int u\,u'\,d\sigma = (u, u'). \tag{8a}$$

In §26 we shall return to a discussion of the connection between this expression and the scalar product in ordinary vector analysis. We call the integral in (8a) the "scalar product" of u and u'.

We first prove the theorem that n continuous, real, mutually orthogonal but otherwise arbitrary functions $u_1, u_2, \ldots, u_\nu, \ldots, u_n$ are *linearly independent*. For if there existed an equation of the form

$$\sum_{\nu=0}^{n} c_\nu u_\nu = 0 \quad \text{with} \quad (u_\mu, u_\nu) = 0 \quad \text{for all} \quad \mu \neq \nu\ ,$$

then by the "scalar multiplication" of the equation by u_μ we would obtain

[2] Courant-Hilbert, Methoden der mathematischen Physik, 2nd ed., Springer, Berlin 1931, Chapter II.

$$c_\mu (u_\mu, u_\mu) = 0 \text{, hence } c_\mu = 0 \text{ for all } \mu,$$

which contradicts the assumption of linear dependence.

We now proceed step by step and first treat the case of *simple degeneracy*. Let u_1, u_2 be continuous, real, not necessarily orthogonal functions belonging to the same eigen value. We consider the family

$$u = c_1 u_1 + c_2 u_2$$

and consider the member which is orthogonal to u_1. This member is given by the condition

$$0 = (u_1, u) = c_1 (u_1, u_1) + c_2 (u_1, u_2).$$

We satisfy this condition by setting

$$c_1 = -(u_1, u_2), \; c_2 = (u_1, u_1) \tag{9}$$

where $c_2 \neq 0$ and hence $u \neq 0$. In u_1 and u we have two mutually orthogonal eigenfunctions of the family, which we choose as the representatives of the family instead of u_1, u_2. We now can normalize u by multiplication with a constant factor such that

$$(u,u) = (u_1,u_1). \tag{9a}$$

For *twofold degeneracy* let u_1, u_2 be two functions that are normalized according to (9) and (9a), and let u_3 be a function of the same eigen value that is not necessarily orthogonal to the first two. We consider the family

$$u = c_1 u_1 + c_2 u_2 + c_3 u_3$$

and select the member of the family that is orthogonal to both u_1 and u_2, thus obtaining the conditions

$$0 = (u_1, u) = c_1 (u_1, u_1) + c_3 (u_1, u_3),$$
$$0 = (u_2, u) = c_2 (u_2, u_2) + c_3 (u_2, u_3).$$

We satisfy both conditions by setting

$$c_1 = -(u_1, u_3), \; c_2 = -(u_2, u_3), \; c_3 = (u_1, u_1) = (u_2, u_2). \tag{10}$$

The functions u_1, u_2, u are mutually orthogonal and hence linearly independent; furthermore we can normalize u so as to obtain:

$$(u,u) = (u_1,u_1) = (u_2,u_2). \tag{10a}$$

We thereby obtain the desired orthogonalization for twofold degeneracy.

This process can obviously be continued in the case of higher degeneracy. The degenerate eigenfunctions are thus made mutually orthogonal; due to (8) they are already orthogonal to the eigenfunctions which belong to different k.

To the orthogonality condition (8) we add the *normality condition*

$$(u,\ u) = \int u^2\, d\sigma = 1 \tag{11}$$

This "normalization to 1" leads to a certain simplification of the orthogonalization process above (see, e.g. (10a)). We shall see in §26 that (11) also has its vector-analytic analog. We still must mention that for complex u equation (11) must be amended to read

$$(u\, u^*) = \int u\, u^*\, d\sigma = 1 \tag{11a}$$

and that in separable problems the normalization is best carried out for each individual factor. Thus in (2) we have to multiply the sine functions by the factors

$$\sqrt{\frac{2}{a}} \quad \text{and} \quad \sqrt{\frac{2}{b}} \quad \text{respectively} \tag{12}$$

and in (3) we have to multiply the exponential function and the Bessel function by the factors

$$\frac{1}{\sqrt{2\pi}} \quad \text{and} \quad \frac{\sqrt{2}}{a\, I'_m(ka)} \quad \text{respectively} \tag{12a}$$

(the latter is due to (20.9a)). Thus our solutions in (2) and (4) at the beginning of this section are determined also with respect to their amplitudes.

From the above-mentioned examples we deduce two theorems concerning nodal lines, which we shall prove now for membranes with arbitrary boundaries:

1. If several nodal lines intersect at a point then they intersect at equal angles (isogonally): for two such lines the angle is $\pi/2$, for ν lines it is π/ν.

2. The larger the eigen value k, the finer the subdivision of the membrane into regions of alternating signs; for $k \to \infty$ the nodal lines become everywhere dense.

In order to demonstrate that theorem 1 holds for our special examples we refer to Figs. 23 and 24, where the boundary itself must be considered a nodal line and the angles are $\pi/2$ and $\pi/4$ as shown. In the case of the full circle we see from (3) that there are m radial lines intersecting at its center at an angle of π/m. In order to show that theorem 2 is satisfied in our cases it suffices to note that for the case of the rectangle and the eigen value k_{nm} the rectangle is subdivided into subrectangles of sides a/n and b/m, so that for $k \to \infty$ at least one side approaches zero.

For the proof of theorem 1 we develop u in the neighborhood of the point O in a Fourier series. We use an r, φ - coordinate system whose origin is at O. For any shape of the membrane we obtain the expansion

$$u = \sum_n I_n(kr)(a_n \cos n\varphi + b_n \sin n\varphi) \tag{13}$$

which converges in a certain neighborhood of O, where the a,b are determined coefficients which can be computed from the given u. The fact that the radial functions in the Fourier expansion must be the Bessel functions I_n follows from differential equation (1) and the regularity of u at O. Now if there is to exist *at least* one nodal line through the point O $(r = 0)$, then according to (13)

$$0 = I_0(0)\, a_0, \text{ and hence } a_0 = 0.$$

Then if a_1 and b_1 are not both zero there is *only* one line through O whose direction is determined by the equation:

$$0 = I_1(kr)\,(a_1 \cos \varphi + b_1 \sin \varphi).$$

Hence for $r > 0$

$$\tan \varphi = -\frac{a_1}{b_1}.$$

This determines the direction of the nodal line *uniquely*.

If there is to be more than one nodal line through O then we must have $a_1 = b_1 = 0$. If we do not at the same time have $a_2 = b_2 = 0$ then according to (13) we have

$$0 = I_2(kr)\,(a_2 \cos 2\varphi + b_2 \sin 2\varphi)$$

or, if there are to be ν nodal lines through O, and hence all a,b up to but not including a_ν, b_ν vanish, then we have

$$0 = I_\nu(kr)\,(a_\nu \cos \nu\varphi + b_\nu \sin \nu\varphi).$$

In the latter case we have for $r > 0$

$$\tan \nu\varphi = -\frac{a_\nu}{b_\nu}. \tag{13a}$$

The right side of this equation is given by our Fourier expansion and shall be denoted by $\tan \alpha$. The general solution of (13a) is then:

$$\varphi = \alpha, \quad \alpha + \frac{\pi}{\nu}, \quad \alpha + \frac{2\pi}{\nu}, \ldots, \alpha + \frac{(\nu - 1)\pi}{\nu}. \tag{13b}$$

These angles differ by the constant amount π/ν, which proves the isogonality.

Passing to the proof of theorem 2, we consider two functions u,v where u is a solution of (1) that satisfies the given boundary condition and v is the special solution

$$v = I_0(kr).$$

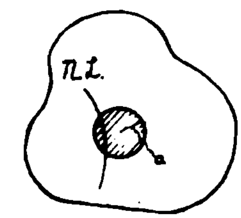


Fig. 25. With increasing k the nodal lines become denser and denser regardless of the shape of the membrane. The proof is given by considering a small disc anywhere on the membrane whose radius a decreases to zero for increasing k. N.L. stands for a nodal line which intersects the disc.

The value of k which is common to u and v is assumed to be large. With the help of this large k we define a small length a by setting $k\,a = \varrho_1$ where ϱ_1 is the first root of the equation $I_0(\varrho) = 0$. We consider a circular disc of radius a situated anywhere on the nodal line pattern of the eigenfunction u (see Fig. 25). With this disc as our domain of integration we apply Green's theorem:

$$\int (u\,\Delta v - v\,\Delta u)\,d\sigma = \int \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right) ds. \tag{14}$$

The left side vanishes since both u and v satisfy the differential equation (1) with the same k. On the right side we have for $r = a$

$$v = 0 \text{ and } \quad \frac{\partial v}{\partial n} = k\,I_0'(\varrho_1) \neq 0.$$

If we set $ds = a\,d\,\varphi$ then equation (14) becomes

$$\varrho_1 I_0'(\varrho_1) \int_0^{2\pi} u\,d\varphi = 0,$$

and hence

$$\int_0^{2\pi} u\,d\varphi = 0. \tag{15}$$

According to this u assumes both positive and negative values on the circumference of the disc. Hence there must be at least two zeros of u on the circumference; that is, our disc must be intersected by at least one nodal line. The disc becomes smaller as k becomes larger and hence for increasing eigen value k the nodal lines become *arbitrarily dense*. This holds for every part of the nodal line pattern.

§ 26. General Remarks Concerning the Boundary Value Problems of Acoustics and of Heat Conduction

The eigenfunctions of the oscillating membrane can be adapted directly to the spatial case. Here we do not think of an oscillating rigid body, but (in order to avoid all complications involving vectors and tensors) rather of an oscillating air mass in the interior of a closed rigid hull of finite extension. Just as on p.166, we interpret the scalar function u as the velocity potential of the air oscillations and we again set the boundary condition $\partial u/\partial n = 0$.

For the rectangular solid with side lengths a,b,c we have, in analogy to (25.2),

$$u = u_{nml} = \cos n\pi\frac{x}{a}\cos m\pi\frac{y}{b}\cos l\pi\frac{z}{c} \tag{1}$$

with eigen value

$$k = k_{nml} = \pi\sqrt{\frac{n^2}{a^2}+\frac{m^2}{b^2}+\frac{l^2}{c^2}}. \tag{1a}$$

This state is non-degenerate if a,b,c are incommensurable.

For a sphere of radius a we obtain the general eigenfunction in analogy to (25.3):

$$u = u_{nml} = \psi_n(k_{nl}r)\, P_n^m(\cos\vartheta)\, e^{im\varphi} \tag{2}$$

Under our boundary condition the eigen value is given by

$$\psi_n'(k_{nl}a) = 0, \tag{2a}$$

where k_n is the l-th root of this equation. This state is $2n$-fold degenerate, since k_{nl} is independent of m and the different states P_n^m for upper index $-n \leq m \leq +n$ belong to the same k_{nl}.

Also in this category are the eigenfunctions of the circular cylinder ($0 < r < a$, $0 < z < h$) which we derived in §20 C. With the boundary condition $\partial u/\partial n = 0$ they are given by

$$u_{nlm} = I_n(\lambda r)\, e^{\pm in\varphi}\cos m\pi\frac{z}{h}; \tag{3}$$

the corresponding eigen value is determined from the equation $I_n'(\lambda a) = 0$, the l-th solution of which we denote by λ_{nl}. Therefore

$$k^2_{nlm} = \lambda^2_{nl} + m^2\pi^2/h^2. \tag{3a}$$

Due to the factor $\exp(\pm i n\varphi)$ in (3) this state is simply degenerate for $n > 0$.

We now consider these eigenfunctions "normalized to 1" where we have to keep in mind the remarks on pp. 173,174 . Then for example in (1) we have to replace $\cos\, n\,\pi\, x/a$ by

$$\cos n\,\pi\,\frac{x}{a}\sqrt{\frac{2}{a}}$$

and according to (22.31b) we have to replace P_n^m in (2) by

$$\Pi_n^m = P_n^m \quad \sqrt{(n+\tfrac{1}{2})\,\frac{(n-m)!}{(n+m)!}},$$

etc. (see exercise V.1).

We now generalize the fundamental theorem on p. 169 and its (mathematically non-rigorous) proof to the case of an arbitrary spatial region S. The theorem now reads: *There exists an infinite system of eigenfunctions*

$$u_1, u_2, \ldots, u_n, \ldots,$$

whose elements are regular in the interior of S and satisfy the differential equation

$$\Delta u_n + k_n^2\, u_n = 0\,,$$

as well as a homogeneous boundary condition. The corresponding eigen values

$$k_1, k_2, \ldots, k_n, \ldots,$$

ordered in an increasing sequence, are infinite in number and increase to infinity; if S is bounded then they form a "discrete spectrum" and they are real since the differential equation was assumed free from absorption.

This system of eigenfunctions satisfies the conditions of *orthogonality* and of *normality:*

(4) $$\int u_n\, u_m\, d\tau = \delta_{n\,m}\,,$$

which according to (25.8a) can be written as

(4a) $$(u_n\,,\, u_m) = \delta_{n\,m}$$

or for complex eigenfunctions

(4b) $$(u_n, u_m^*) = \delta_{n\,m}\,.$$

If the system of u_n is *complete* (see p. 5) then we claim that any continuous point function f given on S can be expanded in the u_n:

(5) $$f = \sum A_n\, u_n\,.$$

If this expansion is possible then, according to (4b), we obtain from (5) through termwise integration

$$A_n = \int f\, u_n^*\, d\tau. \tag{5a}$$

That this expansion is possible is postulated by the *Ohm-Rayleigh principle*, which we shall assume in the following discussion without presenting its mathematical proof. In connection with the name of this principle we remark: Georg Simon Ohm was not only the discoverer of the basic law of Galvanic conduction, but also did profound research in acoustics. He found that the differences in the tone-color of different musical instruments are the result of differences in the mixture of basic tone and overtones. Since, according to (25.1), the overtones ω_n are related to the k_n, and since they are harmonic with the basic tone only for strings and organ pipes, so the construction of an arbitrary tone-color means the construction of an arbitrary function from the (in general anharmonic) eigen values. In Lord Rayleigh's classic book, *Theory of Sound*, this principle is generalized in the sense of equation (5) and is applied in many directions.

We shall now make some remarks about so-called *Hilbert space*, not only to justify the notation (u_n,u_m) of (4a,b) which is reminiscent of vector analysis, but also to give the Ohm-Rayleigh principle an elegant geometric interpretation, which in the hands of the Hilbert school has even been worked out as a means of proving this principle.

In accord with Courant-Hilbert (see p. 172) we define, in a space of N dimensions, the basis vectors $\mathbf{e}_1$, $\mathbf{e}_2 . \ldots , \mathbf{e}_N$ (corresponding to the i,j,k of three-dimensional vector analysis) which lie in the coordinate directions $x_1,x_2, \ldots ,x_N$ and whose scalar product is to satisfy the condition

$$(\mathbf{e}_n\mathbf{e}_m) = \delta_{nm} \tag{6}$$

We further consider a vector which forms the angles $\alpha_1, \alpha_2, \ldots$ with the coordinate axes

$$\mathbf{a} = \cos\alpha_1\, \mathbf{e}_1 + \cos\alpha_2\, \mathbf{e}_2 + \cdots + \cos\alpha_N\, \mathbf{e}_N \tag{7}$$

and we call it a ***unit vector*** if the scalar product of $\mathbf{a}$ with itself has the value 1:

$$(\mathbf{a},\mathbf{a}) = \sum_{n=1}^{N} \cos^2\alpha_n = 1. \tag{7a}$$

A second unit vector b with direction angles β_n is called *orthogonal* to

$\mathbf{a}$ if the scalar product of $\mathbf{a}$ and $\mathbf{b}$ vanishes:

$$(7b) \qquad (\mathbf{a},\mathbf{b}) = \sum_{n=1}^{N} \cos\alpha_n \cos\beta_n = 0.$$

Equations (7a,b) are seen to be generalizations of well-known formulas from three-dimensional analytic geometry.

In the limit $N \to \infty$ we now obtain Hilbert space. Here we note a formal analogy between the basis vectors $\mathbf{e}_n$ and the elements u_n of our system of eigenfunctions. The relations between the latter as written in the form (4a) are formally the same as the relations (6) between the $\mathbf{e}_n$. The system u_n, if it is complete, can serve as substitute for the basis $\mathbf{e}_n$. The same is true for the u_n^* in the case of complex u_n. Every other system of functions that is orthogonalized and normalized to 1 can be composed from the u_n in the sense of equation (7) and can be visualized as a *vector* in Hilbert space. Two such vectors can be transformed into each other by a *rotation* of Hilbert space. But according to (5) any function f is composed of the u_n. With the system of coordinates which is formed by the u_n the function f is associated by (5) to a certain *point* of Hilbert space. The coordinates of this point as measured in the system u_n are the expansion coefficients A_n. Hilbert space thus becomes a *function space*. The association between the arbitrary functions and the points of the space of infinitely many dimensions is one-to-one. If we join the point which represents the function f to the origin of the coordinate system of the u_n, then this infinite dimensional vector represents the function f. According to (5a), which we can write in the form $A_n = (f, u_n^*)$, the coordinates of the representative point are the projections of the representative vector on the axes of the system of u_n^*.

From these highly abstract generalizations we return to the physical applications. For the time being we restrict ourselves to the simple problems of acoustics and heat conduction in their historical form. We defer the questions of wave mechanics to the end of this chapter.

The general problem of *acoustics* for the interior of an arbitrary shell S is the following: the wave equation

$$(8) \qquad \frac{\partial^2 v}{\partial t^2} = c^2 \Delta v, \qquad c = \text{speed of sound},$$

is to be solved with the boundary condition $\partial v/\partial n = 0$ so that for $t = 0$ the functions v and $\partial v/\partial t$ become equal to arbitrary prescribed functions v_0 and v_1 in S. This problem is solved by:

$$(9) \qquad v = \sum A_n u_n \cos\omega_n t + \sum B_n u_n \sin\omega_n t,$$

where the A_n and B_n are to be determined so that

(9a) $$v_0 = \sum A_n u_n \quad \text{and} \quad v_1 = \sum B_n \omega_n u_n$$

Due to the relation of the ω_n with the eigen values k_n (namely $c = \omega_n/k_n$) the second equation can be rewritten as:

(9b) $$\frac{v_1}{c} = \sum B_n k_n u_n.$$

From this we obtain as in (5a)

(9c) $$A_n = \int v_0 u_n^* \, d\tau, \quad B_n = \frac{1}{k_n c}\int v_1, \, u_n^* \, d\tau.$$

We see that this is an *initial value problem;* the *boundary value problem* has been shifted to the u_n.

The general *heat conduction problem* can be solved in the same manner. The difference is that now *one* arbitrary function v_0 suffices to describe the initial state, the initial temperature variation $\partial v/\partial t$ being determined by the differential equation of heat conduction. As a boundary condition we may use any one of the conditions a),b),c) on p. 63, to which we then also subject the eigenfunctions u_n.

We now set

(10) $$v = \sum A_n u_n e^{-\varkappa k^2_n t}$$

where $\varkappa$ stands for temperature (not heat) conductivity. The coefficients A_n are again determined by the initial condition $v = v_0$:

(10a) $$A_n = \int v_0 u_n^* \, d\tau.$$

In addition to this initial condition the function v satisfies the differential equation (25.1) and the boundary condition to which the u_n are subjected.

The potential equation $\Delta u = 0$ has *no* eigenfunctions, or rather every solution which is regular in the interior of S and which satisfies the boundary condition $u = 0$ or $\partial u/\partial n = 0$ must be zero or constant in the interior of S. Hence there can be here no closed "nodal surfaces" $u = 0$ or $\partial u/\partial n = 0$. However, in the next section we shall construct a solution of the general potential boundary value problem (given values $u = U$ on the boundary) from the eigenfunctions of the wave equation.

A solution of the potential equation which is regular in S can also have no maximum or minimum in the interior of S. Extremal values of u can be assumed only on the boundary of S. This follows from Gauss' theorem on the *arithmetic mean* which can be deduced from Green's theorem (see exercise V.2).

Also, *no* eigenfunctions exist for the differential equation $\Delta u - k^2 u = 0$ or the more general $\Delta u - F u = 0$ for positive $F(x,y,z)$ (see exercise II.2).

§ 27. Free and Forced Oscillations. Green's Function for the Wave Equation

The eigenfunctions correspond to free oscillations; in a non-absorbing medium they need no energy supply. We now wish to consider *forced* oscillations, which must be stimulated in the rhythm of their period in order to be able to continue in their purely periodic state. Just as the free oscillations, they are to satisfy a homogeneous surface condition, e.g., $u = 0$; the region S will be assumed to be bounded in the discussions in this section. The measure of stimulation shall, for the time being, be assumed to be a continuous point function in the interior of S, and in analogy to the Poisson equation of potential theory we denote it by ϱ.[3] Correspondingly we write the differential equation of forced oscillations as:

(1) $$\Delta u + k^2 u = \varrho.$$

Here $k = \omega/c$, as we remarked in (26.9a), where ω is the circular frequency of the stimulation and c is the speed of sound. We assume

(2) $$k \neq k_n,$$

i.e., k is different from every eigen value of the region S for the same boundary condition. The case of "resonance" $k = k_n$ will be treated at the end of this section.

According to the Ohm-Rayleigh principle we can expand ϱ in terms of the normalized u_n as in (26.5) and (26.5a):

(3) $$\varrho = \sum A_n u_n, \qquad A_n = \int \varrho\, u_n^* \, d\tau,$$

we also write the solution u of (1) in the same form:

(3a) $$u = \sum B_n u_n.$$

Substituting these expansions in (1) and considering the differential equations $\Delta u_n + k_n^2 u_n = 0$, which differ from (1) and which are satisfied by the eigenfunctions u_n, by equating the coefficients of u_n on both sides we obtain

[3] The function ϱ does not represent charge density as in potential theory, but is of dimension $\sec^{-1}$ if u stands for an acoustic velocity potential.

(4) $$B_n = \frac{A_n}{k^2 - k_n^2}, \qquad u = \sum \frac{A_n u_n}{k^2 - k_n^2}.$$

We now consider the special case in which ϱ is a δ-function[4] and hence the stimulation is limited to a simple source point Q of yield 1 (see §10 C). We then have

$$\int_Q \varrho \, d\tau = 1,$$

for a domain of integration which contains the point Q, and

$$\int \varrho \, d\tau = 0.$$

for a domain of integration which does not contain Q. Hence we obtain from (3)

(4a) $$A_n = u_n^*(Q) \int_Q \varrho \, d\tau = u_n^*(Q)$$

and from (4)

(5) $$G(P, Q) = \sum \frac{u_n(P)\, u_n^*(Q)}{k^2 - k_n^2}.$$

where the u of (4) is now denoted by the more suggestive $G(P,Q)$. Indeed this solution is *Green's function of our differential equation* (1) *for arbitrary positions of the action point P and the source point Q and an arbitrary region S.* We assume only the complete system of eigenfunctions and eigen values for the region S. It should be noted that the Ohm-Rayleigh principle has not been applied to the singular δ- function, but only to the continuous function ϱ of (3), which may, e.g., be taken as a regular Gauss error function. Hence in our derivation we do not need the expansion in terms of the u_n of an arbitrary function but only of certain special everywhere regular functions. In the same manner the termwise differentiation which was needed in the derivation of (4) has been carried out on the regular function (3a) before passage to the limit and not on the limit (5).

Green's function is also the solution of an *integral equation*. In order to demonstrate this we recall equation (10.13a), which holds for every self-adjoint differential expression $L(u)$ and hence in particular for the wave equation $\Delta u + k^2 u$. For the three-dimensional case and the boundary value $u = 0$ it reads:

(6) $$u_Q = \int \varrho(P)\, G(P, Q)\, d\tau_P.$$

[4] We have dropped the name "peak function" ("Zackenfunktion") which was introduced by the author (see *Jahresber. Deutschen Math. Vereinigung* **21**, 312, 1912) in favor of Dirac's notation "δ-function."

The function $G(P,Q)$ is called the *"kernel"* of the integral equation. Corresponding to the reciprocity theorem d) on p. 1 , which, for complex G, has to be rewritten as

$$G(P,Q) = G^*(Q,P), \tag{6a}$$

we call G a "symmetric kernel." From the structure of (5) we see directly that (6a) is satisfied.

The convergence of the series in (5) is absolute only in the one-dimensional case; in the two- or more dimensional case the convergence is conditioned by the alternation of signs of the eigenfunctions for a suitable arrangement of the series. This is the reason equation (5) does not appear explicitly in Hilbert's theory of integral equations, but in an integrated form in which it converges absolutely. In the one-dimensional case equation (5) has been rigorously proven by Erhardt Schmidt.[5]

The non-absolute convergence of (5) becomes apparent if we try to show by termwise differentiation that the differential equation (1) is satisfied. For then we obtain from the n-th term

$$\Delta u_n + k^2 u_n = \Delta u_n + k_n^2 u_n + (k^2 - k_n^2) u_n = (k^2 - k_n^2) u_n$$

and cancelling the factor $k^2 - k_n^2$ with the denominator and summing with respect to n we obtain

$$\Delta G + k^2 G = \sum u_n(P) u_n^*(Q). \tag{6b}$$

For $P = Q$ the sum on the right side consists of positive terms and diverges, as it should; the fact that it converges for $P \neq Q$ and vanishes throughout is caused by the alternating signs and cannot be proven from this representation. The order of increase for $P \to Q$ can be deduced directly from the differential equation (1) as follows. We consider a sphere with small radius r and center Q, and integrate (1) over its interior. Due to the δ-character of ϱ the right side becomes equal to 1. According to Gauss' theorem the first term on the left side becomes

$$\int \frac{\partial G}{\partial r} d\sigma = 4\pi r^2 \frac{\partial G}{\partial r},$$

while the second term vanishes. Hence we have

$$\frac{\partial G}{\partial r} = \frac{1}{4\pi r^2}, \qquad G = -\frac{1}{4\pi r} + \text{Const} \quad \text{for } r \to 0. \tag{7}$$

This expresses the fact that $G(P,Q)$ has a *unit source* in the point $P = Q$.

The above formulas can be interpreted best in Hilbert space (see

[5] In his famous dissertation, Göttingen, 1905.

p. 179). Namely, equation (6b) states that $\Delta G + k^2 G$ is the scalar product of the two unit vectors $u(P)$ and $u^*(Q)$. Hence these unit vectors are orthogonal if $u(P)$ and $u(Q)$ are different ($P \neq Q$); if $u(P)$ and $u(Q)$ are equal ($P = Q$) then orthogonality is of course excluded; instead the product becomes infinite. The expression (5) is constructed from the individual terms of the same product with the "resonance denominator" $k^2 - k_n^2$ as weighting factor.

Despite its poor convergence equation (5) has frequently been found useful in wave mechanical computations (see §30). For the time being we apply it in order to close a gap in the theory of spherical harmonics. But first we make a few preparatory remarks:

1. If the system of eigenfunctions is *separable* then the summation in (5) decomposes into three summations corresponding to the three coordinates. For the rectangular solid we should have:

$$\Sigma = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \sum_{l=1}^{\infty} \tag{8}$$

where n,m,l are as in (26.1).

2. Green's function depends only on the position of the points P,Q relative to the boundary surface σ and on their distance R. It is independent of the orientation of the coordinates in space. A transformation of the coordinate system which transforms the surface σ into itself and leaves R fixed leaves $G(P,Q)$ *invariant*.

3. If σ is the surface of a sphere then the condition of invariance is satisfied for every rotation of the *spherical polar system* r, ϑ, φ with $r = 0$ as the center of the sphere. The coordinates r, ϑ, φ shall be those of P, $r_0, \vartheta_0, \varphi_0$ those of Q.

4. In the latter case we face the additional fact that the system of eigenfunctions (26.2) is *degenerate*, since the eigen value k_n as defined by (26.2a) is independent of m. Writing G as a triple sum in analogy to (8), we can take the denominator $k^2 - k_n^2$ and the radial part of the eigenfunctions in front of the summation over m. Hence we have

$$G(P,Q) = \sum_{n=0}^{\infty} \sum_{l=0}^{\infty} \frac{\Psi_n(k_{nl}\, r)\, \Psi_n(k_{nl}\, r_0)}{k^2 - k_{nl}^2}\, Y_n\,, \tag{9}$$

$$Y_n = \sum_{m=-n}^{+n} \Pi_n^m(\cos\vartheta)\, \Pi_n^m(\cos\vartheta_0)\, e^{im(\varphi - \varphi_0)}\,. \tag{9a}$$

where Ψ_n stands for the function ψ_n of (26.2) normalized to 1, and Π_n for the spherical harmonic P_n nomalized in the same manner. The function Y_n is a surface spherical harmonic. In (9a) we have used the fact that, due to the real character of Ψ_n and Π_n^m the *conjugate* *complex* of the eigenfunction

$$\Psi_n(k_{nl} r)\, \Pi_n^m(\cos\vartheta)\, e^{im\varphi}$$

for the argument $Q = (\ r_0, \vartheta_0, \varphi_0\ )$ can be written as

$$\Psi_n(k_{nl} r_0)\, \Pi_n^m(\cos\vartheta_0)\, e^{-im\varphi_0},$$

for all values of m between $-\ n$ and $+n$.

From remark 2 concerning the invariance of G, and from representation (9), we now see that the surface spherical harmonic (9a) has an invariant meaning which is independent of the rotation of the polar coordinate system. But this is the very theorem which we assumed as an axiom for the proof of the addition theorem of spherical harmonics on p. 133. That proof is now completed.

Up to now we have assumed that stimulation of forced oscillation takes place in the *interior* of the region S. We now wish to assume that stimulation takes place from the *surface*. This is the case if, instead of the homogeneous boundary condition $u = 0$, we prescribe the *inhomogeneous* boundary condition

$$u = U. \tag{10}$$

The surface is then held in pulsation with the rhythm ω of the forced oscillation and with the amplitude U which may vary from point to point, while in the interior of S the differential equation (1) holds throughout with $\varrho = 0$. From (10.12) we know that this boundary value problem can be solved with the help of Green's function by the formula

$$u_Q = \int U \frac{\partial G}{\partial \nu_P}\, d\sigma_P, \tag{11}$$

where the variable of integration on the right side is P and the domain of integration is the surface of S ($d\sigma_P =$ element of surface, $d\nu_P =$ element of normal at the point P). According to (5) equation (11) becomes

$$u_Q = \sum \frac{u_n^*(Q)}{k^2 - k_n^2} \int U\, \frac{\partial u_n(P)}{\partial \nu_P}\, d\sigma_P. \tag{11a}$$

This formula contains the general solution of the famous *Dirichlet problem of potential theory*, for by setting $k = 0$ we obtain

$$u_Q = -\sum \frac{u_n^*(Q)}{k_n^2} \int U\, \frac{\partial u_n(P)}{\partial \nu_P}\, d\sigma_P. \tag{12}$$

The remarkable fact about this solution is that it is not expanded in particular solutions of the differential equation $\Delta u = 0$ concerned, but rather in the eigenfunctions of the *wave equation* (there are no eigenfunctions of the potential equation). Equation (12) remains valid if instead of the boundary condition (10) we prescribe the more general condition

$$\frac{\partial u}{\partial n} + h\,u = U$$

except that in this case we must subject the eigenfunctions u_n to the corresponding homogeneous condition

$$\frac{\partial u}{\partial n} + h\,u = 0$$

In the special case of a sphere of radius a we obtain from (12) and the boundary condition (10)

$$\begin{aligned} \frac{2\pi}{a^2}\, u\,(r_0, \vartheta_0, \varphi_0) \\ = -\sum_n \sum_l \sum_m \frac{A_{nm}}{k_{nl}} \Psi_n(k_{nl} r_0)\, \Psi'_n(k_{nl} a)\, \Pi_n^m(\cos\vartheta_0)\, e^{-im\varphi_0} \end{aligned} \tag{13}$$

$$A_{nm} = \iint U\, \Pi_n^m(\cos\vartheta)\, e^{im\varphi} \sin\vartheta\, d\vartheta\, d\varphi. \tag{13a}$$

where Ψ_n and Π_n have the same meaning as before. The extra factor 2π on the left side of (13) is due to the fact that, as with the Bessel functions and the spherical harmonics, we have to normalize the two functions $\exp(-i\,m\,\varphi_0)$ and $\exp(i\,m\,\varphi)$ to 1.

Written in terms of the same variables $Q = (r_0, \vartheta_0, \varphi_0)$ and expanded in terms of particular solutions of $\Delta u = 0$ our solution reads:

$$2\pi\, u\,(r_0, \vartheta_0, \varphi_0) = \sum_n \sum_m A_{nm} \left(\frac{r_0}{a}\right)^n \Pi_n^m(\cos\vartheta_0)\, e^{-im\varphi_0}. \tag{14}$$

By comparing these solutions we obtain remarkable summation formulas (see exercise V.3).

Finally, we must consider the exceptional case $k = k_m$. From the mechanics and the electrodynamics of oscillating systems we know the "resonance catastrophe": if the rhythm of the stimulating force equals a proper frequency of the system the oscillations increase to infinity. The condition for this event is $\omega = \omega_m$, and hence $k = k_m$. Equation (1) then assumes the form:

$$\Delta u + k_m^2\, u = \varrho\,. \tag{15}$$

Here we have an inhomogeneous equation whose left side coincides with the homogeneous equation of a free oscillation.

For simplicity we first consider the two-dimensional case of the membrane of §25, which now, however, is subjected to a periodically changing transversal pressure[6] $\varrho = \varrho(x, y)$ with an arbitrary distribution over the membrane. Do pressure distributions exist for which the resonance catastrophe is avoided, that is, for which equation (15) has continuous solutions throughout (for the boundary condition $u = 0$)? The answer to this question is physically evident: for such a solution the pressure on the membrane may do *no work*. Hence we must have:

$$\int \varrho\, u_m\, d\sigma = 0. \tag{16}$$

The pressure distribution must be orthogonal to the eigenfunction $u = u_m$ with which it is in resonance, e.g., it may have equal magnitude in oppositely oscillating sectors of the membrane; *in particular the pressure along a nodal line may be of arbitrary strength.*

This orthogonality theorem is a corner stone in the theory of integral equations and has important applications in the perturbation theory of wave mechanics. Here we must be content with uncovering its physical basis.

The orthogonality theorem can be adapted directly to the three-dimensional case if in (16) we replace the surface integral with respect to $d\sigma$ by a volume integral with respect to $d\tau$. Then we see that the expansion coefficients A_n and B_n in (3) and (4) vanish for $n = m$. By passing from the continuous distribution ϱ to a δ-function we obtain information about Green's function in the case of resonance. From $A_m = 0$ and equation (4a) we have $u_m^*(Q) = 0$. In other words: *The singularity of Green's function must lie on a nodal surface of the critical proper oscillation u_m.*

For this position and only for this position of the source point Q an everywhere regular Green's function exists. The special form of Green's function for the case of resonance is obtained from the general form (5) by omitting the term involving k_m; it therefore reads:

$$G(P, Q) = \sum_{n \neq m} \frac{u_n(P)\, u_n^*(Q)}{k_m^2 - k_n^2} \tag{17}$$

§ 28. Infinite Domains and Continuous Spectra of Eigen Values. The Condition of Radiation

With increasing domain the eigen values become closer and closer; for an infinite domain they are dense everywhere; we then deal with a *continuous spectrum of eigen values.*

[6] More precisely: pressure divided by surface tension T (see equation (25.1)). The dimension of ϱ is not that of pressure dyn/cm.^{2}, but $\frac{\text{dyn}}{\text{cm}^2} \Big/ \frac{\text{dyn}}{\text{cm}} = \text{cm}^{-1}$.

Let us consider, e.g., the interior of a sphere of radius a for vanishing boundary values. For the case of purely radial oscillations its eigen values are given by the equation

$$\psi_0 (k_\nu a) = 0, \qquad \psi_0(\varrho) = \frac{\sin \varrho}{\varrho}. \tag{1}$$

Hence $k_\nu a = \nu\pi$ and the difference of successive eigen values is

$$\Delta k_\nu = \frac{\pi}{a} \to 0 \quad \text{for} \quad a \to \infty.$$

We may therefore consider the function $\psi_0 (kr)$ which is everywhere regular and vanishes at infinity as an *eigenfunction of infinite space.* Thus, if we have an acoustical or an optical problem in which the prescribed sources are in the finite domain (with a discrete or a continuous distribution), and which is to be solved for a given wave number k, then we can always add the function ψ_0 to the solution. Hence oscillation problems (in contrast to potential problems) are *not* determined *uniquely* by their prescribed sources in the finite domain. This paradoxical result shows that the condition of *vanishing* at infinity is not sufficient, and that we have to replace it by a stronger condition at infinity. We call it the *condition of radiation:* the sources must be *sources,* not *sinks,* of energy. The energy which is radiated from the sources must scatter to infinity; *no energy may be radiated from infinity into the prescribed singularities of the field* (plane waves are excluded since for them even the condition $u = 0$ fails to hold at infinity).

For our special eigenfunctions

$$\psi_0 (kr) = \frac{1}{2i}\left(\frac{e^{ikr}}{r} - \frac{e^{-ikr}}{r}\right)$$

the state of affairs is simple: for the time dependence $\exp(-i\omega t)$ e^{ikr}/r is a *radiated,* e^{-ikr}/r an *absorbed,* $\psi_0(kr)$ a *standing* wave (nodal surfaces $kr = \nu\pi$). By excluding absorption from infinity we exclude the addition of the eigenfunction $\psi_0(kr)$. Hence the permissible singularities are restricted to the form

$$u = C\,\frac{e^{ikr}}{r} \tag{1a}$$

For these singularities we have the condition

$$\lim_{r \to \infty} r\left(\frac{\partial u}{\partial r} - iku\right) = 0, \tag{2}$$

It is called the general *condition of radiation* and we shall apply it to all acoustic and electrodynamic oscillation problems that are generated by sources in the finite domain.

In fact, condition (2) holds not only for the spherical wave (1a) which radiates from $r = 0$, but it also holds for a stimulation which acts at the point $x = x_0$, $y = y_0$, $z = z_0$

$$u = C\frac{e^{ikR}}{R}, \quad R^2 = (x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2$$

Hence, for a continuous stimulation of the spatial density $\varrho = \varrho(x_0, y_0, z_0)$ we have:

$$u = \int \varrho\frac{e^{ikR}}{R}\,dx_0\,dy_0\,dz_0 \,.$$

This holds not only for unbounded space, but also in the case where there are bounded surfaces σ on which arbitrary linear boundary conditions are prescribed, whether homogeneous, e.g., $u = 0$, or inhomogeneous, e.g., $u = U$. In the former case we have scattered or reflected radiation emanating from the surface σ, whereas in the latter case we have radiation that is stimulated by the pulsating surface σ itself (see p. 186).

As counterpart to the radiation condition (2) we have what may be called the "absorption condition":

$$\text{(2a)} \qquad \lim_{r\to\infty} r\left(\frac{\partial u}{\partial r} + iku\right) = 0 \,.$$

We demonstrate the general validity of the radiation condition by showing that it guarantees the *uniqueness* of solution of the above general oscillation problem. We may then be convinced that the unique solution of the *mathematical* problem is identical with the *solution that is realized in nature*. Our problem is the following:

a) In the exterior of a surface σ, which may consist of several surfaces $\sigma_1, \sigma_2, \ldots$, the function u is to satisfy the differential equation

$$\Delta u + k^2 u = \varrho$$

The function ϱ measures the yield of the sources which may be continuously distributed or concentrated in single points. The function ϱ is given and must vanish at infinity with sufficient rapidity.

b) On σ the function u is to satisfy $u = U$, where U is a given point function on σ. The surface σ lies entirely in the finite domain.

c) In the finite domain u satisfies the condition (2). The quantity r in (2) stands for the distance from any fixed finite point $r = 0$. Around this point we draw a sphere Σ of radius $r \to \infty$, which does not intersect the surface σ. The surface element on the sphere is $d\Sigma = r^2\, d\omega$, where $d\omega$ is the solid angle seen from $r = 0$. The region between Σ and σ is called S.

d) Except at possible prescribed sources the function u is to satisfy those conditions of continuity which we prescribed in the derivation of the differential equation.

We assume that two solutions of this problem u_1 and u_2 exist and, as usual, form

$$w = u_1 - u_2, \tag{3}$$

as well as the conjugate function w^*. These functions satisfy the conditions a) to d) with $\varrho = 0$ and $U = 0$. Then in Green's theorem

$$\int_S (w \Delta w^* - w^* \Delta w)\, d\tau = \left\{\int_\sigma d\sigma + \int_\Sigma r^2 d\omega\right\} \left(w \frac{\partial w^*}{\partial n} - w^* \frac{\partial w}{\partial n}\right) \tag{4}$$

the integral on the left and the first integral on the right vanish. *Hence, the integral over the sphere Σ must also vanish.*

For the further discussion we write:

$$w = \frac{e^{ikr}}{r} \sum_{n=0}^{\infty} \frac{f_n(\vartheta, \varphi)}{r^n}, \tag{5}$$

which is shown to be sufficiently general by the following: we consider w expanded in surface spherical harmonics $Y_n(\vartheta, \varphi)$. According to §24 A the coefficients must be of the form

$$C_n\, \zeta_n^1(k\,r) + D_n\, \zeta_n^2(k\,r)$$

where ζ is connected with the half-index Hankel functions by equation (21.15). But here we must have $D_n = 0$, because of the behavior of ζ_n^2 for large values of the argument (see §21 D, p. 117). At the same place we learned that the ζ_n^1 are composed of a finite number of terms of the form $e^{ikr}/(k\,r)^m$, $m < n$. By arranging this expansion in spherical harmonics according to powers of r^{-n} we obtain (5), where the $f_n(\vartheta, \varphi)$ turn out to be finite sums of surface spherical harmonics.

The f_n satisfy a simple *recursion formula.* According to (22.4) the differential equation $\Delta w + k^2 w = 0$ written in terms of r, ϑ, φ, yields the equation

$$\frac{\partial^2 (r\,w)}{\partial r^2} + \frac{1}{r^2} D(r w) + k^2\, r\, w = 0\,. \tag{6}$$

where D is the differential symbol of (23.15b) in the coordinates ϑ, φ. Applying (6) to (5) we obtain

$$e^{ikr} \sum_{n=0}^{\infty} \left(- \frac{2ikn}{r^{n+1}} + \frac{n(n+1)}{r^{n+2}} + \frac{D}{r^{n+2}} \right) f_n = 0 .$$

Replacing the index of summation n in the first term of the parentheses by $n + 1$ we obtain

$$e^{ikr} \sum_{n=0}^{\infty} \frac{1}{r^{n+2}} [- 2ik(n+1) f_{n+1} + \{n(n+1) + D\} f_n] = 0$$

and hence the recursion formula:

$$2ik(n+1) f_{n+1} = \{n(n+1) + D\} f_n . \tag{6 a}$$

Hence: *if* $f_0 = 0$ *then all* $f_1 = f_2 = \ldots = 0$.

We now investigate the remaining integral in (4). Since we are interested in the limit for $r \to \infty$ we can replace w by the first term of its expansion (5), ignoring the higher powers of $1/r$, whence:

$$w = \frac{e^{ikr}}{r} f_0 , \qquad w^* = \frac{e^{-ikr}}{r} f_0^* ;$$

$$\frac{\partial w}{\partial n} = ik \frac{e^{ikr}}{r} f_0 , \qquad \frac{\partial w^*}{\partial n} = - ik \frac{e^{-ikr}}{r} f_0^* .$$

Thus we obtain:

$$\int r^2 d\omega \left(w \frac{\partial w^*}{\partial n} - w^* \frac{\partial w}{\partial n} \right) = - 2ik \int f_0 f_0^* d\omega .$$

The integrand is *positive* as long as $f_0 \neq 0$. But we saw in (4) that this integral must vanish. Hence

$$f_0 = 0, \text{ and due to (6a) } f_1 = f_2 = \ldots = 0.$$

Therefore

$$w = 0 \text{ and } u_2 = u_1.$$

The author's original proof[7] of this uniqueness theorem assumed, in addition to the conditions a),b),c) for u, the existence of Green's function for the exterior of the surface and an additional "finality condition." The fact that the latter is superfluous has been rigorously proven by F. Rellich[8] even for the case of an arbitrary number of dimensions h

[7] See footnote on p. 183 and Frank-Mises II, chap. XIX, §5. The form of the proof given in the text is essentially F. Sauter's.

[8] *Jahresber Deutschen Math. Vereinigung* **53**, 57 (1943), which also treats the case in which the surface σ stretches to infinity.

where the radiation condition reads

$$\lim_{r\to\infty} r^{\frac{h-1}{2}}\left(\frac{\partial u}{\partial r} - i\,k\,u\right) = 0. \tag{7}$$

In the two-dimensional case $h = 2$, where, as we know, the spherical wave e^{ikr}/r is replaced by the cylindrical wave $H_0^1(kr)$, equation (7) becomes

$$\lim_{r\to\infty} r^{\frac{1}{2}}\left(\frac{\partial u}{\partial r} - i\,k\,u\right) = 0, \tag{7a}$$

which actually is satisfied by $u = H_0^1(kr)$. In the one-dimensional case, where the radiating wave is given by $\exp(i\,k\,|x|)$, equation (7) becomes

$$\lim_{|x|\to\infty} \left(\frac{\partial u}{\partial |x|} - i\,k\,u\right) = 0. \tag{7b}$$

Following Rellich, we stress the fact that no radiating solution u of the wave equation can exist which, in every direction, approaches zero more rapidly than $1/r$. For such a function u we would have $f_0 = 0$ in (5) and, as we have seen, this causes u to vanish identically. In this respect the wave equation differs from the potential equation. For the latter solutions exist which, for increasing r, decrease more rapidly than $1/r$, the so-called dipole, quadrupole, and octupole fields of §24 C. For the wave equation such an r-dependence, which implies a pole of higher order than $1/r$ at $r = 0$, can happen only in the so-called "near zone" ($r < \lambda$, $\lambda =$ wavelength); in the "far zone" ($r > \lambda$) every solution of the wave equation behaves like the spherical wave e^{ikr}/r. Potential theory is the limiting case $\lambda = \infty$, as for this case, the near zone reaches to infinity, so to speak.

We now come to the problem of *Green's function for a continuous spectrum.* We first consider in detail the very simplest one-dimensional example ($-\infty < x < +\infty$), in which the radiation condition is the only boundary condition prescribed. Green's function is then identical with the "principal solution" introduced on p. 47, and therefore has a "unit source" at an arbitrary prescribed point $x = x_0$ (see exercise II.3). It must satisfy the conditions:

a) $$\frac{d^2G}{dx^2} + k^2\,G = 0 \quad \text{for } x \neq x_0$$

b) $$\left(\frac{dG}{dx}\right)_{x_0+0} - \left(\frac{dG}{dx}\right)_{x_0-0} = 1, \quad \text{(definition of unit source)}$$

c) $$\frac{dG}{d|x|} - i\,k\,G = 0 \quad \text{for } x = \pm\infty.$$

The solution is seen to be

$$(8) \qquad G = \begin{cases} \dfrac{1}{2\,i\,k}\, e^{ik(x-x_0)} & \text{for} \quad x > x_0\,, \\[2mm] \dfrac{1}{2\,i\,k}\, e^{-ik(x-x_0)} & \text{for} \quad x < x_0\,. \end{cases}$$

We compare this to the representation (27.5) first for the finite region $-l < x < +l$, but with the usual boundary conditions replaced by the radiation condition. In preparation for a continuous spectrum we change the name k_n of the eigen values to λ ; the eigenfunction $u = u_\lambda$ which belongs to λ is then defined by

$$\text{a)} \qquad \frac{d^2u}{dx^2} + \lambda^2 u = 0 \qquad -l < x < +l\,,$$

$$\text{b)} \qquad \frac{du}{d|x|} - i k u = 0 \qquad |x| = l\,.$$

If we write the solution of a) as:

$$(9) \qquad u = A\, e^{i\lambda x} + B\, e^{-i\lambda x}\,,$$

then according to b) we must have

$$A(\pm\,\lambda - k)\, e^{\pm i\lambda l} + B(\mp\,\lambda - k)\, e^{\mp i\lambda l} = 0\,,$$

$$\frac{A}{B} = \frac{\lambda + k}{\lambda - k}\, e^{-2i\lambda l} = \frac{\lambda - k}{\lambda + k}\, e^{+2i\lambda l}.$$

From this we obtain the equation for λ:

$$\left(\frac{\lambda - k}{\lambda + k}\right)^2 e^{4i\lambda l} = 1\,.$$

This equation splits into the equations

$$(9\text{a}) \qquad \frac{\lambda - k}{\lambda + k}\, e^{2i\lambda l} = +1, \quad B = A, \quad u = 2\,A \cos \lambda x,$$

$$(9\text{b}) \qquad \frac{\lambda - k}{\lambda + k}\, e^{2i\lambda l} = -1, \quad B = -A, \quad u = 2\,i\,A \sin \lambda x.$$

From (9a) we obtain as first and second approximation

$$(10\text{a}) \qquad \begin{array}{ll} \text{for } \lambda \gg k,\ \lambda = \dfrac{\pi}{l}\, m & \text{and } = \dfrac{\pi}{l}\, m \left(1 - \dfrac{ikl}{\pi^2 m^2}\right), \quad m \to \infty\,, \\[2mm] \text{for } \lambda \ll k,\ \lambda = \dfrac{\pi}{l}\,(m + \tfrac{1}{2}) & \text{and } = \dfrac{\pi}{l}\,(m + \tfrac{1}{2})\left(1 - \dfrac{i}{kl}\right),\ m = 0, 1, 2, . \end{array}$$

where m is an integer. In the same manner we obtain from (9b)

$$\text{(10b)}\qquad \begin{aligned} &\text{for } \lambda > k,\ \lambda = \frac{\pi}{l}(m+\tfrac{1}{2}) \text{ and } = \frac{\pi}{l}(m+\tfrac{1}{2})\left[1-\frac{ikl}{\pi^2(m+\tfrac{1}{2})^2}\right], m\to\infty,\\ &\text{for } \lambda < k,\ \lambda = \frac{\pi}{l}m \qquad \text{and } = \frac{\pi}{l}m\left(1-\frac{i}{kl}\right),\ m = 0, 1, 2, \ldots \end{aligned}$$

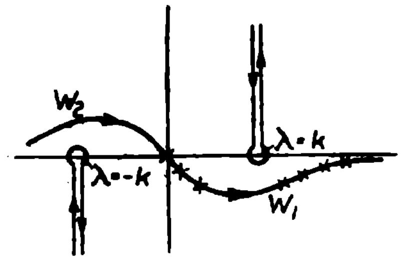


Fig. 26. The path of integration W_1 is completed by the path W_2 to an infinite closed path $W = W_1 + W_2$. For positive $x - x_0$ this path can be deformed so that it runs in the positive imaginary λ- half plane.

We see that the values of λ as calculated from (9a) and (9b) form a sequence (marked by x in Fig. 26) that, starting with $\lambda = 0$, first descends linearly into the negative imaginary λ- half plane[9] and finally for large λ (large m) osculates the real λ- axis from below. According to (9a,b) the successive points alternatingly belong to cosine and sine eigenfunctions. After normalization to 1 these eigenfunctions are

$$\text{(11)}\qquad u = \begin{cases} \dfrac{1}{\sqrt{l(1+\Lambda)}}\cos\lambda x, \\[2mm] \dfrac{1}{\sqrt{l(1-\Lambda)}}\sin\lambda x, \end{cases} \qquad \Lambda = \frac{\sin\lambda l\cos\lambda l}{\lambda l}.$$

In the limit $l\to\infty$ the λ- points of Fig. 26 will be everywhere dense on the right half curve denoted by W_1. The difference between two successive points of the sequence (10a) or (10b) then always becomes

$$\text{(11a)}\qquad d\lambda = \frac{\pi}{l} \to 0.$$

We now return to the representation (27.5) of Green's function. For $u(P)$ and $u(Q)$ we substitute their expression (11) in the variables x and x_0 respectively, and combine the pairs of successive cosine and sine terms, i.e., the terms which belong to successive eigen values λ. The numerator of (27.5) then becomes

$$u(P)u^*(Q) = \frac{\cos\lambda x\cos\lambda x_0}{l(1+\Lambda)} + \frac{\sin\lambda x\sin\lambda x_0}{l(1-\Lambda)}.$$

According to (11) and (11a) we have for $l\to\infty$

[9] The fact that the eigen values are complex in contrast to the theorem on p. 169 is due to the fact that our present boundary condition is of a complex nature.

$$\Lambda = 0, \qquad \frac{1}{l} = \frac{d\lambda}{\pi}.$$

Hence the numerator in (27.5) becomes $\cos\lambda\,(x - x_0)\,d\lambda/\pi$, while the denominator in our present notation is $k^2 - \lambda^2$. Equation (27.5) then becomes

$$G = \frac{1}{\pi}\int\limits_{W_1} \frac{\cos\lambda\,(x - x_0)}{k^2 - \lambda^2}\,d\lambda = \frac{1}{2\pi}\int\limits_{W} \frac{e^{i\lambda\,(x - x_0)}}{k^2 - \lambda^2}\,d\lambda. \tag{12}$$

where W in the last term is the path $W_1 + W_2$ of Fig. 26. The fact that the integration over W_1' is equal to one half the integral over the whole path W follows from the fact that in the integral over W_1 both the numerator and the denominator are even functions of λ. The fact that in the last term we can replace the cosine by the exponential function follows from fact that the sine part of the exponential function is odd in λ and hence vanishes upon integration. The path W is much more convenient than W_1 since it can be deformed away from the origin by the methods of complex integration.

The manner in which this deformation should be performed can be seen from Fig. 26. For positive $x - x_0$ the path W can be drawn over into the positive imaginary λ half plane, for negative $x - x_0$ it can be drawn into the negative imaginary half plane. In the one case the path can not be transformed across the pole $\lambda = +k$ of the integrand in (12), in the other case it can not be transformed across the pole $\lambda = -k$. Forming the residues and combining the two cases we obtain from (12):

$$G = \frac{1}{2ik}\,e^{ik|x - x_0|}. \tag{13}$$

This is exactly the same as (8).

Hence we see: *The general representation (27.5) of Green's function remains valid for a continuous eigen value spectrum if, in accordance with the radiation condition, we consider a complex path of integration.* If instead we have the "*absorption condition*" (i replaced by $-i$ in (1a) and (2)), then instead of W we have to consider its reflected image on the real λ-axis; we then obtain equation (13) with i replaced by $-i$.

If instead of the one-dimensional case we consider the two- or three-dimensional case and correspondingly replace the coordinate x by the polar coordinates r, φ and r, ϑ, φ, then the spectrum of the eigen values becomes *continuous only in the r-coordinate* but remains *discontinuous* in the angle coordinates. For example in the case of unbounded three-dimensional space we start from the following representation of Green's function

(14) $$2\pi G(P,Q) = \sum_n \sum_m \Pi_n^m (\cos\vartheta)\, \Pi_n^m (\cos\vartheta_0)\, e^{im(\varphi-\varphi_0)} \int_{W_1} \frac{F\, d\lambda}{k^2-\lambda^2},$$

(14 a) $$F = \Psi_n(\lambda r)\; \Psi_n(\lambda r_0).$$

Here, as in the preceding section, Π and Ψ stand for the spherical harmonic and Bessel functions normalized to 1; and in the following the Z^1, Z^2 correspond to the Hankel functions ζ^1, ζ^2. The factor 2π on the left side is due to the normalization of the functions $\exp\{im\varphi\}$ and $\exp\{-im\varphi_0\}$. As in Fig. 26 the path W_1 lies in the complex λ-plane from $\lambda = 0$ to $\lambda = \infty$. and again avoids the pole $\lambda = k$ We first give a brief discussion of the way in which this representation can be treated in analogy to the one-dimensional case. This will yield a representation of spherical and cylindrical waves which we have met before.

In order to transform W_1 into the path W of Fig. 26, we write

$$\Psi_n(\lambda r) = \tfrac{1}{2}(Z_n^1(\lambda r) + Z_n^2(\lambda r)),$$

For the convergence problems which arise in connection with the normalization we refer the reader to Appendix I. Due to the properties of Hankel functions (see exercise IV.2, in particular equation (12), and also the discussion in connection with equation (32.13)), we can transform the integral over W_1, which involves the function F of (14a), into the integral over W involving

(14b) $$F_1 = \tfrac{1}{2} Z_n^1(\lambda r)\; \Psi_n(\lambda r_0) \qquad r > r_0,$$

and

(14 c) $$F_2 = \tfrac{1}{2} \Psi_n(\lambda r)\; Z_n^1(\lambda r_0) \qquad r < r_0$$

Since the integrand $\frac{1}{2}F_{1,2}/(k^2-\lambda^2)$ vanishes at infinity in the positive imaginary λ- plane for both cases $r \gtrless r_0$, the integral of (14) reduces to the residue at the pole $\lambda = k$:

(15) $$\int F_{1,2} \frac{d\lambda}{k^2-\lambda^2} = \frac{\pi}{2ik} \begin{cases} Z_n^1(kr)\; \Psi_n(kr_0) & r > r_0, \\ \Psi_n(kr)\; Z_n^1(kr_0) & r < r_0. \end{cases}$$

Applying the addition theorem of spherical harmonics (22.34) we obtain from (14):

$$(16)\qquad G(P,Q)=\frac{1}{4ik}\sum_{n=0}^{\infty}\Pi_n(\cos\Theta)\,\Pi_n(1)\begin{cases}Z_n^1(kr)\,\Psi_n(kr_0) & r>r_0,\\ \Psi_n(kr)\,Z_n^1(kr_0) & r<r_0.\end{cases}$$

For reasons of symmetry $G(P,Q)$ in unbounded space is a pure function of the distance

$$R=\sqrt{r^2+r_0^2-2rr_0\cos\Theta},$$

between P and Q; namely, due to the definition of the unit source on p. 47 we have

$$(16\,\text{a})\qquad G(P,Q)=-\frac{1}{4\pi}\frac{e^{ikR}}{R}=\frac{1}{4\pi i}\zeta_0^1(kR),$$

where ζ_0 is given by (21.15a). If, on the right side of (16), we pass from Π, Ψ, Z to P, ψ, ζ (see Appendix I equation (9a)), we obtain the addition theorem (24.9a).

The corresponding series for the two-dimensional case are contained in (21.3).

More important than the derivation of these known formulas is the generalization to the case in which space is not unbounded but is bounded by a finite closed surface σ (or, in the two-dimensional case, by a curve s) with prescribed boundary conditions. We are then dealing with the *proper problem of Green's function:* to find a function $G(P,Q)$ having a unit source in Q, satisfying the radiation condition at infinity and the given boundary condition on σ (or s).

We choose the special case in which the surface σ is a sphere $r = a$, and the boundary condition is

$$(17)\qquad u = O.$$

The point Q is to lie on the ray

The eigenfunction which belongs to the eigen value λ is no longer $\psi_n(\lambda r)$, but can be written in the (non-normalized) form

$$(18)\qquad u_n(\lambda,r)=\psi_n(\lambda r)+A\,\zeta_n^1(\lambda r)$$

Due to (17) the function A becomes[10]

$$(18\text{a})\qquad A=-\frac{\psi_n(\lambda a)}{\zeta_n^1(\lambda a)}.$$

For the construction of Green's function we shall not follow the general method of equation (14). Instead we shall use a shorter though

[10] The fact that A depends on λ made it necessary to write $u(\lambda,r)$, instead of $u(\lambda r)$.

less systematic approach based on equation (24.9) for unbounded space:

$$\left.\begin{array}{l}(19a)\\(19b)\end{array}\right\} \quad \frac{e^{ikR}}{ikR} = \sum_{n=0}^{\infty} (2n+1) P_n(\cos\vartheta) \begin{cases} \psi_n(kr_0)\,\zeta_n^1(kr) & r > r_0, \\ \zeta_n^1(kr_0)\,\psi_n(kr) & r < r_0. \end{cases}$$

Here (19b) will not yet satisfy condition (17) for $r = a$; in order to satisfy (17) we complete the right side of (19b) by adding

$$-\sum_{n=0}^{\infty} (2n+1) P_n(\cos\vartheta)\, \zeta_n^1(kr_0)\, \psi_n(ka) \frac{\zeta_n^1(kr)}{\zeta_n^1(ka)},$$

Due to (18) the right side of (19b) becomes

$$\sum_{n=0}^{\infty} (2n+1) P_n(\cos\vartheta)\, \zeta_n^1(kr_0)\, u_n(k, r). \tag{20}$$

If we make the same adjunction to (19a) then the continuous passage from (19a) to (19b) for $r = r_0$ is preserved, as is the radiation condition for $r \to \infty$. The right side of (19a) becomes

$$\sum_{n=0}^{\infty} (2n+1) P_n(\cos\vartheta)\, \zeta_n^1(kr)\, u_n(k, r_0). \tag{21}$$

From (20) and (21) we obtain Green's function by adjoining the factor $k/4\pi i$ which, as in (16a), is due to the condition of a unit source. We then have:

$$G(P, Q) = \frac{k}{4\pi i} \sum_{n=0}^{\infty} (2n+1) P_n(\cos\vartheta) \begin{cases} \zeta_n^1(kr)\, u_n(k, r_0) & r > r_0, \\ \zeta_n^1(kr_0)\, u_n(k, r) & r < r_0. \end{cases} \tag{22}$$

This way of writing reveals the connection with our general method in (14). The function F of (14a) is now represented by

$$F = u_n(\lambda, r)\, u_n(\lambda, r_0);$$

except for a constant normalizing factor. The corresponding functions F_1, F_2 of (14b,c) become

$$F_{1,2} = \frac{1}{2} \begin{cases} \zeta_n^1(\lambda r)\, u_n(\lambda, r_0) & r > r_0, \\ \zeta_n^1(\lambda r_0)\, u_n(\lambda, r) & r < r_0 \end{cases}$$

By forming the residues for $\lambda = k$ we then obtain equation (22).

In Appendix II of this chapter we shall introduce a novel method of constructing Green's function, which not only improves the convergence of the series in the most important cases, but also reveals new aspects of the method's applicability.

In the appendix to the following chapter we shall further show that this method would solve the problem of wireless telegraphy on a spherical earth (for infinitely conductive soil and a vertical "dipole antenna") if it were not for the decisive role of the ionosphere.

Finally we remark: a representation of the form (14) remains valid if as the surface σ we choose an ellipsoid instead of a sphere. Instead of the r, ϑ, φ we then have to use the coordinate system of confocal ellipsoids and hyperboloids. The spectrum of eigen values for the exterior of the ellipsoid will then remain discrete in the parameters of the one piece and two piece hyperboloids, but becomes continuous in the parameter of the ellipsoids. By integration over this last parameter we would obtain a simplification similar to that of (22). Even in the most general case where there are no separating coordinates, in which the eigenfunctions can be decomposed into products, we can still use equation (27.5) as a starting point for the representation of Green's function.

§ 29. The Eigen Value Spectrum of Wave Mechanics. Balmer's Term

The Schrödinger equation of wave mechanics for the simple case of the hydrogen atom reads

$$\Delta\psi + \frac{2m}{\hbar^2}(W - V)\,\psi = 0\,. \tag{1}$$

This is our equation (7.15), with the difference that the symbol of energy W has been replaced by the difference of the total energy W and the potential energy, V or, mechanically speaking, by the kinetic energy. The Rutherford model for the H-atom consists of a nucleus, the proton with a $+e$ charge, and of an electron with a $-e$ charge that moves in the proton field. Its potential (Coulomb) energy measured in electrostatic units is

$$V = -\frac{e^2}{r}, \tag{2}$$

where r is the distance from the proton and V is normalized so that at infinity we have $V = 0$. The mass energy m_0c^2 of the electron at rest is not to be counted in the total energy. In the following we may consider the proton at rest at the point $r = 0$.

Equation (1) differs from the wave equation we have treated so far because the constant k^2 has been replaced by a *point function* which becomes *singular* at the point $r = 0$. Whereas we have used k to denote eigen value, we shall now use W as an eigenparameter. Hence, we shall seek those values of W for which (1) has a solution which is continuous in the entire space. These solutions are the eigenfunctions of our "Kepler problem," where the nucleus plays the role of the sun and the electron the role of the planets. Since the electron may move in unbounded space, the spectrum of eigen values will be continuous in the r-coordinate as in equation (28.14). More important for us is the fact that the spectrum also has *discrete* components.

The spectral apparatus gives us the discrete spectrum by measuring the *line spectrum*, which, in the case of hydrogen, is given in the visible range by the Balmer series H_α, H_β, H_γ, H_δ, . . . The lines of this spectrum cumulate at the limit given by the Rydberg constant R. The adjoining *continuum* lies in the near ultraviolet range. Both the discrete and the continuous spectrum are given by the Schrödinger equation. This equation reduces to a simple mathematical formula the enigma of the spectral lines, with their finite cumulation point, the behavior of which differs so fundamentally from that of all mechanical systems.

Niels Bohr gave a general explanation of the Balmer series and its limiting frequencies twelve years before Schrödinger, by endowing the Rutherford model with certain quantum theoretical traits. However the concept of orbits he used lead to diverse contradictions and had to be abandoned in favor of the analytic model of equation (1). The fact that (1) is also based on quantum theory is indicated by the entrance of Planck's constant $\hbar = h/2\pi$.

What is the physical meaning of the eigenfunction ψ ? The answer to this question shows the complete revolution in the concept of nature that quantum theory has brought about: $|\psi|^2 dx\, dy\, dz$ stands for the *probability* with which we may expect to find the hydrogen electron at the point (x,y,z) within an error of dx,dy,dz. Hence, in wave mechanics the concept of probability takes the place of the concept of strict determinism which rules in classical mechanics. The measure of indeterminacy in the atomic range is Planck's h (Heisenberg).

The "normalization of the eigenfunctions to 1," which so far had been introduced only for mathematical simplicity, thereby acquires a fundamental meaning. Namely, the equation

$$\int |\psi|^2 \, d\tau = 1 \tag{3}$$

asserts the *certainty* that the electron is somewhere in space; this condition is necessary from the point of view of wave mechanics. Equation (3) holds for a discrete spectrum; for the continuum it must be modified according to the prescription of Appendix I to this chapter.

We now turn to the integration of (1), introducing the coordinates r, ϑ, φ. If we write the wave equation in the form (22.4) and let[11]

$$\psi = \chi(r)\, P_l^m(\cos\vartheta)\, e^{im\varphi} \tag{4}$$

then according to the differential equation (22.13) we obtain

$$\frac{d^2\chi}{dr^2} + \frac{2}{r}\frac{d\chi}{dr} + \left\{\frac{2m}{\hbar^2}\left(W + \frac{e^2}{r}\right) - \frac{l(l+1)}{r^2}\right\}\chi = 0. \tag{5}$$

We first consider the case in which the electron is *tied* to the nucleus. Then W must be negative since the energy of the electron at rest at infinity is normalized to zero. If it is absorbed by the nucleus and stably tied there then its energy is decreased. If, on the other hand, $W > 0$ then even for an infinite distance from the nucleus the electron has positive kinetic energy and, mechanically speaking, has a hyperbolic orbit.

The asymptotic behavior of χ for $r \to \infty$ is obtained from (5) by neglecting all terms with $1/r$ and $1/r^2$:

$$\frac{d^2\chi}{dr^2} + \frac{2m}{\hbar^2} W\chi = 0.$$

For negative W we write

$$\frac{d^2\chi}{d\varrho^2} = \frac{1}{4}\chi, \quad \chi = e^{-\varrho/2}, \quad \varrho = \frac{2r}{\hbar}\sqrt{-2mW}. \tag{5a}$$

The other solution of (5a), namely, $\chi = \exp(+\varrho/2)$, must be neglected since χ is to be finite everywhere.

In order to obtain an exact solution of (5) we write

$$\chi = e^{-\varrho/2}\, v(\varrho) \tag{6}$$

and obtain as the differential equation for v

$$v'' + \left(\frac{2}{\varrho} - 1\right)v' + \left[\frac{n-1}{\varrho} - \frac{l(l+1)}{\varrho^2}\right]v = 0 \tag{6 a}$$

with the abbreviation

$$n = \frac{m e^2/\hbar}{\sqrt{-2mW}}\,. \tag{6 b}$$

[11] Here we denote the lower index of P by l instead of n, corresponding to wave mechanical usage: l = azimuthal quantum number, n_r = radial quantum number, $n = n_r + l + 1$ = total quantum number, n = magnetic quantum number, where we now have $-l \leq m \leq +l$.

In order to discuss (6a) we use the method of equation (19.36). We write

(7) $$v = \varrho^{\lambda} w, \qquad w = a_0 + a_1 \varrho +$$

and in analogy with (19.37) we obtain:

(7a) $$\lambda(\lambda + 1) = l(l + 1), \text{ and hence } \lambda = +l.$$

The other root of (7a) $\lambda = -l - 1$ must be excluded, since v as well as χ must remain finite for $\varrho = 0$. The recursion formula for the a_k is obtained in analogy with (19.37a) by equating to zero the coefficients of $\varrho^{\lambda + k - 1}$ in the power series obtained from (6a) and (7). Thus we find:

(7 b) $$a_{k+1}\,[(\lambda + k + 1)(\lambda + k) + 2(\lambda + k + 1) - l(l + 1)] + a_k\,[n - 1 - \lambda - k] = 0\,.$$

If in this equation we make the coefficient of a_k equal to zero by setting

(8) $$n = k + \lambda + 1,$$

then a_{k+1} vanishes and so do all the subsequent terms in w: *the series breaks off, that is, w becomes a polynomial* of degree k, whose further properties we shall treat later. For the time being we shall stress only the following facts: 1. Due to the factor $\exp(-\varrho/2)$ in (6), we see that as $r \to \infty$ the function χ tends to zero with sufficient rapidity to make possible the normalization of ψ according to (3), no matter what the degree of the polynomial w. 2. If the series did not break off then from (7b) we should obtain an asymptotic behavior of a_k which would make w become infinite to the order $\exp(+\varrho)$ for $\varrho \to \infty$, and the normalization of ψ would be impossible. Hence the requirement that the series for w break off is a wave mechanical *necessity.*

We now consider equation (8). We denote the value of k there by n_r (radial quantum number) and for λ we substitute its value from (7b) (azimuthal quantum number). Hence, according to (8) n is *integral:*

(8 a) $$n = n_r + l + 1\,.$$

This number n is called the "total quantum number." From equation (6b) we obtain:

(8 b) $$W = W_n = -\frac{m e^4}{2 \hbar^2 n^2}\,.$$

Setting W equal to the energy quantum $h\nu$ we obtain

(9) $$\nu = \frac{m e^4}{2 h \hbar^2 n^2} = \frac{R}{n^2}$$

where

(9 a) $$R = \frac{2 \pi^2 m e^4}{h^3}.$$

This R is the above mentioned "Rydberg frequency"; it can be measured spectroscopically with extraordinary precision and hence can lead to an improvement of our knowledge of the fundamental constants e, m, h. The number ν of (9) is called the *Balmer term.*

The observable frequency of a spectral line is obtained by the passage of the atom from an initial state 1 to a final state 2 and is computed as the difference of the associated terms ν_2 and ν_1. Hence for the hydrogen spectrum we have

(10) $$\nu = \nu_2 - \nu_1 = R\left(\frac{1}{n_2^2} - \frac{1}{n_1^2}\right).$$

The Balmer series corresponds to the passage into the final state $n_2 = 2$; the Lyman series which lies in the ultraviolet range corresponds to the passage into the fundamental state of the hydrogen atom $n_2 = 1$; in both cases the passage is from an arbitrary initial state $n_1 > n_2$. Hence we have

(10a) $$\nu = R\left(\frac{1}{2^2} - \frac{1}{n^2}\right), \qquad n = 3, 4, 5, \ldots \qquad \text{Balmer series,}$$

(10b) $$\nu = R\left(\frac{1}{1^2} - \frac{1}{n^2}\right), \qquad n = 2, 3, 4, \ldots \qquad \text{Lyman series.}$$

The series with $n_2 = 3$, $n_2 = 4, \ldots$ lie in the infrared domain.

After having learned about the *eigen values* of the H-atom we wish to consider the analytic character of its *eigenfunctions.* With the use of (7), (7a) and (8a) we obtain from (6a)

(11) $$\varrho w'' + [2(l+1) - \varrho] w' + (n - l - 1) w = 0.$$

This equation is obtained through $(2l + 1)$-fold differentiation from the simpler differential equation

(12) $$\varrho L'' + (1 - \varrho) L' + \mu L = 0 \quad \text{with} \quad \mu = n + l.$$

For every integer μ this equation has one and only one polynomial solution of degree μ. With a suitable normalization we obtain the solutions:

$$\begin{array}{ll}\mu = 0, & L = 1,\\ \mu = 1, & L = -\varrho + 1,\\ \mu = 2, & L = \varrho^2 - 4\varrho + 2,\\ \mu = 3, & L = -\varrho^3 + 9\varrho^2 - 18\varrho + 6.\\ \vdots & \vdots \end{array}$$

These are precisely the expressions we denoted in exercise I.6 as Laguerre polynomials; equation (12) is the Laguerre differential equation, as indicated by the choice of the letter L. This differential equation coincides with the differential equation (24.29) of the confluent hypergeometric function for the parameters $\alpha = -\mu = -n - l$. Hence we have

(13) $$L = F(-n-l, 1, \varrho) \qquad \text{and} \qquad w = \frac{d^{2l+1}L}{d\varrho^{2l+1}}.$$

Hence from (4), (6), (7) and (7a) we obtain the representation of the hydrogen eigenfunction

(14) $$\psi = N\varrho^l e^{-\varrho/2} \frac{d^{2l+1}L}{d\varrho^{2l+1}} P_l^m(\cos\vartheta)\, e^{im\varphi}.$$

where N is a normalization factor due to (3). From (5a) and (8b) we obtain ϱ:

(14a) $$\varrho = \frac{2r}{na}, \qquad a = \frac{\hbar^2}{me^2} \sim \frac{1}{2}10^{-8}\,\text{cm}.$$

where, as is customary, a denotes the "hydrogen radius."

In order to justify this notation, and as a single special application of the above, we compute the "probability density" in the "fundamental state" $n = 1$ of the H-atom. For $n = 1$ we have according to (8a) $l = m = 0$, $n_r = 0$ and hence from (14)

$$\psi = -N_1 e^{-\varrho/2} = -N_1 e^{-r/a}, \qquad |\psi|^2 = N_1^2 e^{-2r/a},$$

where from (3) we obtain $N_1 = (\pi a^3)^{-\frac{1}{2}}$. Hence, the probability of finding the electron is distributed spherically over the nucleus. For $r = 0$ this probability assumes its maximum N_1^2, for $r = a$ its value is only $(N_2/e)^2$, but it only vanishes at infinity. The charge density is proportional to this probability. From the point of view of wave mechanical statistics we do not have an electron which is concentrated at a point, but instead we have a *charge cloud* whose principal part is in the interior of a sphere of radius a.

From the older point of view of orbits we must ascribe a *disc-like* form to the H-atom. The fundamental state (circular orbit of radius a) then corresponds to a circular disc. In a magnetic field all the circular discs of an H-atom gas would have to be parallel to each other and perpendicular to the magnetic force lines; a light ray passing through this gas would have to show "magnetic double-diffraction." Precise measurements by Schütz, though performed not on an H-atom gas but on the analogous Na-vapor, showed no trace of this phenomenon. This is one of the contradictions which have been cleared up by wave mechanics.

A behavior similar to that of the fundamental state of the H-atom is obtained for all states with $l = 0$, the so-called "s-terms" of spectroscopy. For $l = 0$ we obtain from (14)

$$\psi = N_n e^{-\varrho/2} L_n'(\varrho), \qquad \varrho = \frac{2r}{na}, \qquad n = n_r + 1,$$

which again means spherical symmetry. Such s-terms are the fundamental states of the alkali atoms Li, Na, K, ... The same holds for all completed shells, e.g., the so-called eight-shells of rare gases. The proof is based on the addition theorem of spherical harmonics. This spherical symmetry of the closed shells is obviously of great importance for all chemical applications.

We have to add a few remarks about the continuous spectrum of hydrogen, that is, about the states $W > 0$ (the "hyperbolic orbits" of the older theory). The electron is then no longer tied to the nucleus but is still in the field of the proton.

According to (5a) and (6b) ϱ and n become *purely* imaginary for $W > 0$. In the asymptotic solution (5a) the two signs of ϱ are equivalent; both solutions $\exp\{\pm\varrho/2\}$ can be used. It is unnecessary, and due to the imaginary character of ϱ it is also impossible, to make the series (7) break off. Hence every value of W is permissible. *The W-spectrum becomes continuous* and reaches from $W = 0$ to $W = \infty$. Since, according to (8b) $W = 0$ corresponds to the limit $n = \infty$ of the discrete spectra, we see that to each of these spectra there adjoins a continuous spectrum in the short wave direction. The analytic form of the representation (14) remains valid; but L is now no longer a Laguerre polynomial but a *confluent hypergeometric series which does not break off*, since the parameter $\alpha = -n - l$ in (13) is no longer negative integral but general complex.

§ 30. Green's Function for the Wave Mechanical Scattering Problem. The Rutherford Formula of Nuclear Physics

Nuclear physics originated with Rutherford's experiments on the scattering of α-rays by heavy atoms. Since the electron shells of the atom are immaterial for the case of α-rays, we may treat the scattering problem in terms of the continuous H-spectrum. We are dealing, in fact, with a *two-body problem:* a *nucleus* (of charge Ze, where Z is the atomic number, $Z = 1$ for the H-spectrum) and a particle interacting with it (in this case an α-particle with mass m_α and charge $Z'e$ where $Z' = 2$; in the preceding case an electron of mass m and charge $-e$ corresponding to the charge number $Z' = -1$). First we want to find that point of the continuous spectrum that corresponds to the energy constant W_α of the incoming α-rays. For an infinite distance between the α-particle and the nucleus the kinetic energy of the α-particle is

$$W_\alpha = \frac{m_\alpha}{2} v^2, \qquad \text{hence} \qquad 2\, m_\alpha W_\alpha = (m_\alpha v)^2 = p^2,$$

where $\vec{p} = m_\alpha \vec{v}$ is the kinetic momentum of the α-particle.

If we now pass from the *corpuscular interpretation* of the α-rays to the "complementary" *wave interpretation*, then $p/\hbar$ is, at the same time, the *wave number*[12] k_α of the α-rays.

Hence we have

$$(1) \qquad k_\alpha = \frac{m_\alpha v}{\hbar} = \sqrt{\frac{2\, m_\alpha W_\alpha}{\hbar^2}}\,.$$

We can, therefore, rewrite the variable ϱ of (29.5a) in the form

$$(2) \qquad \varrho = 2\, i\, k_\alpha\, r\,.$$

For an arbitrary point of the continuous spectrum (i.e., for an arbitrary value W different from W_α) we replace k_α by λ as in §28. Equations (1) and (2) then generalize to

$$(2a) \qquad \lambda = \sqrt{\frac{2\, m_\alpha W}{\hbar^2}}, \qquad \varrho = 2\, i \lambda r\,.$$

If, as before, we assume the nucleus at rest then the wave equation (29.1) becomes

$$(3) \qquad \Delta\psi + \frac{2\, m_\alpha}{\hbar^2}\left(W - \frac{ZZ'e^2}{r}\right)\psi = 0\,.$$

For the time being we replace (3) by:

[12] In fact the formula $k_\alpha = p/\hbar$ is the equation of L. de Broglie: "h times the reciprocal of the wavelength equals the momentum," which in turn is the relativistic completion of Planck's equation: "h times the reciprocal of the time of oscillation equals the energy."

(3a) $$\Delta\psi + K^2\psi = 0, \qquad K^2 = \lambda^2 - \frac{2\, m_\alpha\, ZZ' e^2}{\hbar^2 r};$$

for the point $\lambda = k_\alpha$ of the spectrum we then have

(3b) $$K_\alpha^2 = k_\alpha^2 - \frac{2\, m_\alpha\, ZZ' e^2}{\hbar^2 r}.$$

We note the important fact that in the difference $K^2 - K_\alpha^2$ the potential term, which is a function of position, is eliminated, so that this difference becomes *independent of position:*

(4) $$K^2 - K_\alpha^2 = \lambda^2 - k_\alpha^2.$$

The reader should convince himself that all our previous deductions from Green's theorem, such as the orthogonality of the eigenfunctions in §26 or the representation of Green's function for *constant* k^2 in §27, remain valid for our generalized wave equation $\Delta\psi + K^2\psi = 0$ with K^2 a *function of position* given by (3a,b).

We now return to the Rutherford scattering experiment. If we consider the source of the α- rays (the radium particle) to be point-like and in the finite domain, then we have a *spherical wave* of corpuscular rays, which is modified by the presence of the nucleus in the manner prescribed by the wave equation (3). However if we remove the source to infinity, which is more natural and at the same time simpler, then we have to treat the same problem for the *plane wave.* In both cases the solution is given by Green's function of §28; in the first case for a general position of the source point Q, in the second case for the limit $Q \to \infty$. Since Green's function was to be summed over the complete system of eigenfunctions, we have to consider the discrete as well as the continuous eigen value spectrum for a finite Q. However in the limit $Q \to \infty$ $u(Q)$ vanishes for all eigen values of the discrete spectrum; hence, in this case we have to carry out the integration over only the continuous spectrum. We may retain the expression (29.14) for the eigenfunctions $u(P)$ in question, if we replace ϱ by $2i\lambda r$ in accordance with (2a). If in addition we let $\vartheta = 0$ be in the direction of the line which joins Q to the position O of the nucleus, then the scattering problem becomes symmetric with respect to the axis $\vartheta = 0$, and hence independent of φ, so that the eigenfunctions $u(P)$ must be independent of φ. Therefore according to (29.14) we have

(5) $$u(P) = \chi_l(\varrho)\, P_l(\cos\vartheta), \quad \chi_l = N\varrho^l e^{-\varrho/2} \frac{d^{2l+1} L}{d\varrho^{2l+1}}.$$

The corresponding expression for $u(Q)$ is obtained from (5) by replacing $P_l(\cos\vartheta)$ by $P_l(\cos\vartheta_0) = P_l(-1) = (-1)^l$, and $\chi_l(\varrho)$ by $\chi_l(\varrho_0)$

and then passing to the limit $\varrho_0 \to \infty$. We then obtain the representation of the plane wave from (28.14). By performing the integration over the path W of Fig. 26 and forming the residue at the pole $\lambda = k_a$ we obtain a series representation of the form

$$\sum C_l \chi_l(\varrho) P_l(\cos\vartheta), \quad \varrho = 2ik_a r, \tag{6}$$

where the coefficients C_l are determined in a somewhat cumbersome fashion in terms of the normalizing factors of the χ and P and of the asymptotic behavior of $\chi(\varrho_0)$. This representation was first derived by W. Gordon.[13]

A much simpler representation is obtained if we replace the polar coordinates r, ϑ by the *parabolic coordinates* ξ, η. We thus obtain as the wave function of the scattering process (see Appendix III of this chapter):

$$\psi = e^{ikx} L_n(ik\eta), \quad \eta = r - x = r(1 - \cos\vartheta). \tag{7}$$

Here k is the wave number of (1)

$$k = k_a = \frac{m_a v}{\hbar}; \tag{7a}$$

and n is the total quantum number, which, becomes purely imaginary, for the continuous spectrum as mentioned at the end of §29. This total quantum number is computed from (29.13) where, as in equation (3), we have to replace e^2 by $-ZZ'e^2$:

$$n = \frac{ie^2 ZZ'}{\hbar v}. \tag{7b}$$

The function L_n is the confluent hypergeometric function of (29.13) for $l = 0$:

$$L_n(\varrho) = F(-n, 1, \varrho). \tag{7c}$$

The variable η is the parabolic coordinate defined in (7), the other coordinate is $\xi = r + x = r(1 + \cos\vartheta)$. In the following ϑ will be called the "scattering angle."

From (7) we obtain the asymptotic value for $r \to \infty$

$$\psi = C_1 e^{ikx} + C_2 \frac{e^{ikr}}{r} \tag{8}$$

with the abbreviations

[13] *Z. Physik* (1928). See also the excellent book by Mott and Massey, The Theory of Atomic Collisions, Oxford, 1933, chapter III.

(8a) $$C_1 = \frac{(-i k \eta)^n}{\Gamma(n+1)}, \qquad C_2 = C_1^* \frac{i n/k}{1-\cos\vartheta}.$$

The first term on the right side of (8) represents the incoming *plane wave*, the second term represents the *spherical wave* scattered from the nucleus. The quantities C_1, C_2 are not constants but depend on η; however, since n is purely imaginary only their phases depend on η. We are interested only in the absolute value of the ratio C_2/C_1, which is independent of η and hence of r, and depends only on the scattering angle ϑ. Namely, from (8a) and (7a,b) we obtain

(9) $$\left|\frac{C_2}{C_1}\right| = \frac{|in|}{k(1-\cos\vartheta)} = \frac{e^2 Z Z'}{m_\alpha v^2(1-\cos\vartheta)} = \frac{e^2 Z Z'}{2 m_\alpha v^2 \sin^2\vartheta/2}.$$

According to the wave mechanical definition (29.3) of probability density, the square of this quantity is the *ratio of the number of scattered particles per unit of spatial scattering angle and the number of incoming particles per unit of area on a surface perpendicular to the incoming direction.* This law was deduced by Rutherford through geometric consideration of the classical hyperbolic orbits without the help of quantum theory. This was possible owing to the fact that the constant $\hbar$ canceled in (9). Rutherford's law holds not only for α- rays but also for any other particles (protons, electrons, . . .) which are in Coulomb interaction with the nucleus, of course with a correspondingly different meaning of Z' and m_α. The interesting "exchange effect" that occurs for the equality of scattering and the scattered particle will not be discussed here. For very high velocities of the scattering particle we should have to use relativity theory.

Appendix I

Normalization of the Eigenfunctions in the Infinite Domain

In passing from a bounded to an unbounded domain we encounter certain difficulties in convergence which can be removed only by a change in the normalization process. This modified process was introduced by H. Weyl in the theory of integral equations and has been adapted to wave mechanics.

As an example we choose the function $I_\nu(kr)$, where ν is arbitrary and k is a root of the equation $I_\nu(ka) = 0$. According to (20.19) its normalizing integral would be:

(1) $$N = \int_0^a [I_\nu(kr)]^2\, r\, dr = \frac{a^2}{2}[I_\nu'(ka)]^2;$$

Due to the asymptotic behavior of $a I'_\nu$ this integral becomes divergent in the limit $a \to \infty$ (undetermined between the limits $\pm\infty$). In order to obtain a normalization of I_ν for $a \to \infty$ we start from the more general integral

$$N' = \lim_{a\to\infty} \int_0^a I_\nu(k\,r)\, I_\nu(k'\,r)\, r\, dr \tag{1a}$$

According to (21.9a) N' behaves like the function $\delta(k|k')$:[14] namely, it vanishes for $k \neq k'$ and becomes infinite for $k = k'$ so that

$$\int_\Delta N'\, k'\, dk' = 1 \tag{2}$$

where Δ is an arbitrary interval containing the critical point $k' = k$. Since in particular Δ can be chosen arbitrarily small, so that k' can be considered constant in Δ, we may replace (2) by

$$\int_\Delta N'\, dk' = \frac{1}{k}, \tag{2a}$$

and similarly

$$\int_\Delta N' \sqrt{k'}\, dk' = \frac{1}{\sqrt{k}}. \tag{2b}$$

We now change the normalizing integral (1) in the limit $a \to \infty$ to

$$N = \int_0^\infty r\, dr\, I_\nu(k\,r) \int_\Delta dk'\; I_\nu(k'\,r), \tag{3}$$

that is, in (1) we replace one[15] of the two proper oscillations I_ν by the group of neighboring proper oscillations

$$\int_\Delta dk'\; I_\nu(k'\,r) \tag{3a}$$

and thus eliminate the above indeterminacy by averaging, so to speak, by interference within the wave group. From (1a) and (2a) we then obtain

$$N = \int_\Delta N'\, dk' = \frac{1}{k}. \tag{4}$$

In order to make clear the mathematical essence of the above

[14] The symbols σ, r, s of (21.9a) have been replaced here by r, k', k.

[15] According to the original method of Weyl we could also replace *both* eigenfunctions I by "*eigendifferentials*" of the form (3a). Instead of the expressions "group of proper oscillations" or "wave group" used in the text, the less attractive term "wave bundle" is customarily used in wave mechanics.

process, we were somewhat careless in the interchanging of the limiting processes $a \to \infty$ and $\Delta \to 0$ (or in other words $k' \to k$). It would therefore be desirable to deduce equation (4) once more on the basis of Green's theorem. Writing

$$(5) \qquad u = I_\nu(k r) e^{i\nu\varphi}, \quad v = I_\nu(k' r) e^{-i\nu\varphi}, \quad d\sigma = r\,dr\,d\varphi, \quad ds = a\,d\varphi$$

we compute[16] $\int (u\Delta v - v\Delta v)\,d\sigma$ in the known manner both as a surface integral and by Green's theorem as a contour integral. We then obtain:

$$(5a) \qquad (k^2 - k'^2)\int_0^a u\,v\,r\,dr = a\left(u\frac{\partial v}{\partial r} - v\frac{\partial u}{\partial r}\right)_{r=a}.$$

If we divide by $k^2 - k'^2$ and integrate with respect to k' (under the integral sign) from $k' = k - \Delta/2$ to $k' = k + \Delta/2$, and then pass to the limit $a \to \infty$, the left side becomes the normalizing integral N of (3). On the right side we choose a so large that we can compute the I_ν asymptotically. If we choose half the sum of (19.55) and (19.56), the constants $\mp(\nu + \frac{1}{2})\pi/2$ in the exponents partly cancel and partly are of no consequence in the following consideration, and so may be omitted. Then we obtain as the right side of (5a):

$$(5b) \qquad \frac{1}{2\pi}\int_{k-\Delta/2}^{k+\Delta/2} \frac{1}{\sqrt{k k'}}\frac{dk'}{k^2 - k'^2}\{(e^{ika} + e^{-ika})\,i k'(e^{ik'a} - e^{-ik'a}) - (e^{ik'a} + e^{-ik'a})\,i k(e^{ika} - e^{-ika})\}.$$

After multiplying out and collecting terms we obtain

$$(6) \qquad \frac{1}{\pi}\int_{k-\Delta/2}^{k+\Delta/2} \frac{dk'}{\sqrt{k k'}}\left\{\frac{\sin(k-k')a}{k-k'} + \frac{\sin(k+k')a}{k+k'}\right\}.$$

For sufficiently small Δ we can replace $\sqrt{k k'}$ by k and put it in front of the integral. If in the first term of the integral we make the substitution $x = (k - k')a$, then it becomes

$$-\int_{x_1}^{x_2} \sin x \frac{dx}{x} \quad \text{with} \quad \begin{cases} x_1 = a\Delta/2, \\ x_2 = -a\Delta/2, \end{cases}$$

In the limit $a \to \infty$ this assumes the value π (see exercise I.5). In the second term of the integral (6) we make the substitution $y = (k + k')a$.

[16]The domain of integration of Green's theorem is not the complete circle of radius a, but the domain of periodicity of u and v as in equation (25.5), namely a circular sector of angle $\alpha = 2\pi/\nu$. In the integration with respect to φ we obtain a factor α on both sides of (5a) that, in the text, has already been canceled.

The limits of integration then become

$$\left.\begin{aligned} y_1 &= (2k - \Delta/2)a \\ y_2 &= (2k + \Delta/2)a \end{aligned}\right\} \to \infty \quad \text{for} \quad a \to \infty.$$

It is easy to see that this second term vanishes. Hence (6) becomes $1/k$ and (5a) becomes

$$N = \frac{1}{k} \tag{7}$$

which coincides with (4). From this it follows that the Bessel function which is normalized to 1 in the above manner is given by

$$\frac{1}{\sqrt{N}} I_\nu(kr) = \sqrt{k}\, I_\nu(kr). \tag{7a}$$

From the relation (21.11)

$$I_{n+\frac{1}{2}}(kr) = \sqrt{\frac{2kr}{\pi}}\, \psi_n(kr) \tag{8}$$

we further see that the function ψ_n normalized to 1, which we denoted by Ψ_n on p. 197, is related to ψ_n by

$$\Psi_n(kr) = \sqrt{\frac{2}{\pi}}\, k\psi_n(kr). \tag{8a}$$

Indeed, from (7a) and (8) we have

$$1 = \int_0^\infty r\,dr \sqrt{k}\, I_{n+\frac{1}{2}}(kr) \int_\Delta dk' \sqrt{k'}\, I_{n+\frac{1}{2}}(k'r) = \frac{2}{\pi}\int_0^\infty r^2\,dr\, k\psi_n(kr) \int_\Delta dk'\, k'\psi_n(k'r)$$

substituting this in (8a) we obtain

$$1 = \int_0^\infty r^2\,dr\; \Psi_n(kr) \int_\Delta dk'\, \Psi_n(k'r), \tag{9}$$

which according to (3) means normalization to 1.

Equation (8) relates not only $I_{n+\frac{1}{2}}$ and ψ_n, but also the associated functions $H^{1,2}_{n+\frac{1}{2}}$ and $\zeta^{1,2}_n$; hence equation (8a) holds also for the ζ_n normalized to 1 which we denoted by Z_n on p. 197.

For a general three-dimensional eigenfunction equation (3) holds in the form

$$N = \int d\tau\, u_n \int_\Delta dk'\, u_n', \tag{10}$$

where u_n and u_n' are the eigenfunctions belonging to the eigen values k and k'. The eigenfunctions normalized to 1 are then

$$U_n = u_n/\sqrt{N}. \tag{10a}$$

Appendix II

A New Method for the Solution of the Exterior Boundary Value Problem of the Wave Equation Presented for the Special Case of the Sphere

The "exterior boundary value problem" consists of the construction of a solution of the wave equation $\Delta u + k^2 u = 0$, which is continuous throughout the exterior of the given bounded surface σ, assumes arbitrarily prescribed boundary values $u = U$ on σ, and satisfies the radiation condition at infinity. We know that this solution is best represented by Green's function which vanishes on σ, satisfies the radiation condition at infinity, and at a prescribed point Q has a discontinuity of the character of a unit source.

In the case of a sphere of radius a and a source point Q with the coordinates $r = r_0$, $\vartheta = 0$, we constructed G in the form of equation (28.22):

$$(1) \quad G = \frac{k}{4\pi i}\begin{cases}\sum C_n\, u_n(k, r_0)\, \zeta_n(k\, r)\, P_n(\cos\vartheta) & \text{for } r > r_0,\\ \sum C_n\, \zeta_n(k\, r_0)\, u_n(k, r)\, P_n(\cos\vartheta) & \text{for } r < r_0,\end{cases} \quad C_n = 2n+1.$$

The radiation condition is satisfied by the factor $\zeta_n(k\,r)$, or more precisely $\zeta_n^1(k\,r)$, in the first line, the boundary condition for $r = a$ is satisfied by the factor in the second line

$$(1\text{b}) \qquad u_n(k, r) = \psi_n(k\,r) + A\,\zeta_n(k\,r), \qquad A = -\frac{\psi_n(k\,a)}{\zeta_n(k\,a)}.$$

where n is a positive integral and hence $P_n(\cos\vartheta)$ is continuous for all values $0 \leq \vartheta \leq \pi$.

We now attempt to solve this problem in a more economical manner, by subjecting $\zeta_\nu(k\,r)$ not only to the radiation condition, but also to the boundary condition

$$(2) \qquad \zeta_\nu(k\,a) = 0$$

Then ν must be integral, since the roots of $\zeta_\nu(\varrho) = 0$ coincide with those of $H^1_{\nu+\frac{1}{2}}(\varrho) = 0$. According to (21.41) these roots are non-integral and *complex* (of large absolute value). We denote the consecutive roots which lie in the first quadrant of the complex ν-plane by $\nu_1, \nu_2, \ldots$ and the general root by ν_m. We use Σ to denote summation over the complete system of these roots, and write

$$(3) \qquad G = \sum D_\nu\, \zeta_\nu(k\,r)\, P_\nu(-\cos\vartheta).$$

The function P_ν here is *not a Legendre polynomial* but the *hyper-*

geometric series of (24.24a)

$$(3a) \qquad P_\nu(-\cos\vartheta) = F\left(-\nu, \nu+1, 1, \frac{1+\cos\vartheta}{2}\right).$$

The fact that in (3) we used $P_\nu(-\cos\vartheta)$ instead of $P_\nu(+\cos\vartheta)$ is due to the fact that G is to be regular on the ray $\vartheta = \pi$ whereas the ray $\vartheta = 0$ is to contain the singular point Q. According to (3a) the function $P_\nu(-\cos\vartheta)$ for $\vartheta = \pi$ has the value $F(-\nu, \nu+1, 1, 0) = 1$; for $\vartheta = 0$ we obtain from (24.32), after replacing ζ by $-\zeta = -\cos\vartheta$, the value

$$(4) \qquad F(-\nu, \nu+1, 1, 1) \rightarrow \frac{\sin\nu\pi}{\pi} \log\vartheta^2 .$$

We now turn to the determination of the coefficients D_ν, to which F. Sauter has made such valuable contributions. We remark: *the functions ζ_ν are mutually orthogonal, that is, we have*

$$(5) \qquad \int_{ka}^{\infty} \zeta_\nu(\varrho)\, \zeta_\mu(\varrho)\, d\varrho = 0, \quad \mu \neq \nu$$

This is a consequence of the differential equation for ζ_ν, which in analogy to (21.11) reads:

$$\varrho \frac{d^2 \varrho\zeta_\nu}{d\varrho^2} + [\varrho^2 - \nu(\nu+1)]\, \zeta_\nu = 0.$$

If we also write the same equation for ζ_μ and multiply these equations by ζ_μ and ζ_ν respectively, then by integrating the difference of these equations over the fundamental domain $ka \leq \varrho < \infty$ we obtain:

$$(5a) \qquad \{\nu(\nu+1) - \mu(\mu+1)\} \int_{ka}^{\infty} \zeta_\mu\, \zeta_\nu\, d\varrho = \varrho\left(\zeta_\mu \frac{d\varrho\zeta_\nu}{d\varrho} - \zeta_\nu \frac{d\varrho\zeta_\mu}{d\varrho}\right)\Bigg|_{ka}^{\infty}.$$

The right side of (5a) vanishes at the lower limit on account of (2), at the upper limit on account of the asymptotic behavior of the ζ according to (19.55). Thus (5) is proven. At the same time (5a) yields the normalizing integral

$$N = \int_{ka}^{\infty} [\zeta_\nu(\varrho)]^2\, d\varrho = \lim_{\mu\to\nu} \varrho \, \frac{\zeta_\mu \frac{d\varrho\zeta_\nu}{d\varrho} - \zeta_\nu \frac{d\varrho\zeta_\mu}{d\varrho}}{\nu(\nu+1) - \mu(\mu+1)}\Bigg|_{ka}^{\infty}$$

Differentiating the numerator and denominator with respect to μ and considering (2) we obtain

$$(6) \qquad N_\nu = \frac{(ka)^2}{2\nu+1}\left(\frac{\partial\zeta_\nu}{\partial\nu}\, \frac{d\zeta_\nu}{d\varrho}\right)_{\varrho=ka},$$

Introducing the abbreviations

$$(6\text{a}) \qquad \eta_\nu(\varrho) = \frac{\partial \zeta_\nu(\varrho)}{\partial \nu}, \qquad \zeta'_\nu(\varrho) = \frac{d\zeta_\nu(\varrho)}{d\varrho}.$$

we can rewrite (6) in the shorter form

$$(6\text{b}) \qquad N_\nu = \frac{(k a)^2}{2\nu+1} \eta_\nu(k a) \zeta'_\nu(k a).$$

If we now multiply (3) by $\zeta_{\bar\nu}(k r)\, k\, dr$ and integrate from $r = a$ to $r = \infty$ then, on account of the orthogonality and the normalization, we obtain:

$$(7) \qquad D_{\bar\nu} N_{\bar\nu} P_{\bar\nu}(-\cos\vartheta) = \int_a^\infty G\, \zeta_{\bar\nu}(k r)\, k\, dr.$$

where $\bar\nu$ is any one of the roots ν.

However this determination of the D is not yet satisfactory since we do not know G. We therefore perform the following limit process: we divide (7) by $P_\nu(-\cos\vartheta)$ (we now omit the bars over the ν) and then let ϑ approach zero. We also split off the source point singularity from G by writing

$$(7\text{a}) \qquad -4\pi G = \frac{e^{ikR}}{R} + g, \qquad R^2 = r^2 + r_0^2 - 2 r r_0 \cos\vartheta.$$

Then equation (7) becomes

$$(7\text{b}) \qquad -4\pi D_\nu N_\nu = \lim_{\vartheta\to 0} \int_a^\infty \left(\frac{e^{ikR}}{R} + g\right) \zeta_\nu(k r)\, k\, dr \Big/ P_\nu(-\cos\vartheta).$$

Now g remains finite whenever $a < r < \infty$ and ϑ is arbitrary, whereas $P_\nu(-\cos\vartheta)$ becomes infinite as $\vartheta \to 0$; hence the contribution of g on the right side of (7b) vanishes. For the same reason we may restrict the contribution of e^{ikR}/R to an integration over the immediate neighborhood of the source point coordinate r_0 by writing

$$r = r_0(1+\eta), \qquad -\varepsilon < \eta < +\varepsilon, \qquad dr = r_0\, d\eta,$$
$$\zeta_\nu(k r) = \zeta_\nu(k r_0), \qquad e^{ikR} = 1,$$

while the denominator R is approximated by:

$$R = r_0 \sqrt{(1+\eta)^2 + 1 - 2(1+\eta)\left(1 - \frac{\vartheta^2}{2}\right)} \sim r_0 \sqrt{\eta^2 + \vartheta^2}.$$

Thus (7b) becomes

(7c) $$4\pi D_\nu N_\nu = -k\,\zeta_\nu(k r_0) \lim_{\vartheta \to 0} \int_{-\varepsilon}^{+\varepsilon} \frac{d\eta}{\sqrt{\eta^2+\vartheta^2}} \Big/ P_\nu(-\cos\vartheta).$$

The limit on the right side is known from (24.31) to be $\pi/\sin\nu\pi$. Hence with the help of (6) we obtain from (7c) the completely determined value

(8) $$D_\nu = -\frac{1}{4\,k a^2} \frac{2\nu+1}{\sin\nu\pi} \frac{\zeta_\nu(k r_0)}{\eta_\nu(k a)\,\zeta_\nu'(k a)}.$$

which no longer depends on ε. Equation (3) for Green's function then assumes the form

(9) $$G = -\frac{1}{4\,k a^2} \sum_\nu \frac{2\nu+1}{\sin\nu\pi} \frac{\zeta_\nu(k r_0)\,\zeta_\nu(k r)}{\eta_\nu(k a)\,\zeta_\nu'(k a)} P_\nu(-\cos\vartheta).$$

This formula becomes considerably simpler if, instead of the ζ, we introduce the normalized eigenfunctions

$$Z_\nu(kr) = \frac{\zeta_\nu(kr)}{\sqrt{N_\nu}}, \qquad Z_\nu(kr_0) = \frac{\zeta_\nu(k r_0)}{\sqrt{N_\nu}}$$

Indeed, with the help of (6b) we can then rewrite (9) as

(9a) $$G = -\frac{k}{4} \sum_\nu Z_\nu(k r_0)\,Z_\nu(kr) \frac{P_\nu(-\cos\vartheta)}{\sin\nu\pi}.$$

Equation (9a) will prove useful later on; for the time being we shall use equation (9), which has the following advantages and disadvantages:

1. In (9) Green's function is represented by the *same* formula both for $r > r_0$ and for $r < r_0$, not by two *different* formulas as in (1).

2. The general requirement of the reciprocity of Green's function is satisfied owing to the fact that (9) is symmetric in r and r_0. On the other hand in equation (1) the reciprocity of G was expressed by the fact that by interchanging r and r_0 we interchange the two representations for $r > r_0$ and $r < r_0$. The reciprocity of G with respect to the angles ϑ and ϑ_0 (we considered the case in which the latter is zero), can be expressed both in (9) and in (1) by replacing $\cos\vartheta$ by $\cos\Theta$ which is symmetric in ϑ and ϑ_0.

3. The orthogonality relation (5) is essentially different from our previous formulations: in (5) ζ_ν is multiplied by ζ_μ and not by ζ_μ^* as in (25.11a); also, in (5) $\zeta_\nu\,\zeta_\mu$ is multiplied by the one-dimensional interval dr not by $r^2 dr$ as in the application of Green's theorem in exercise V.1b.

4. It is remarkable that our representation (9) seems to break

down for $\vartheta = 0$, since then according to (4) every term of the series becomes infinite like $\log \vartheta^2$, whereas the function it represents is to be regular for $\vartheta = 0$ and $r \neq r_0$. Hence the *whole* ray $\vartheta = 0$(not only the *point* $\vartheta = 0, r = r_0$ on it) must be considered a *singularity* of the representation (9). Hence in the neighborhood of this singularity, that is to say in a more or less narrow cone around this ray, our representation will become more or less useless. The question of the completion of our representation for the interior of such a cone shall be postponed to p. 221.

5. On the other hand equation (9) simplifies for the neighborhood of $\vartheta = \pi$, say for $\vartheta = \pi - \delta$, where in the original form (3) it reads:

$$G = \sum D_\nu \zeta_\nu (k r) P_\nu (\cos \delta). \tag{10}$$

We now let r_0 increase to ∞, that is to say we pass from the primary *spherical* wave G to a *plane* wave u coming in the direction $\vartheta = 0$. Then for $H_{\nu+1/2} (k r_0)$ we may use Hankel's approximation (19.55) no matter what the index ν since now the argument is large compared to the index. Then according to (21.15) we may write:

$$\zeta_\nu (k r_0) = \frac{1}{k r_0} e^{i(k r_0 - (\nu+1)\pi/2)}. \tag{10a}$$

By combining all the factors which are independent of ν into the amplitude A we obtain from (8),

$$D_\nu = A \frac{2\nu + 1}{\sin \nu \pi} e^{-i\nu\pi/2} \Big/ \eta_\nu (k a) \zeta_\nu' (k a). \tag{10b}$$

Substituting this value in (10) we obtain the diffraction field of the plane wave u in the rear of a sphere of radius a under the angle of diffraction δ. For the time being we set the boundary condition $u = 0$ for $r = a$; later on we shall discuss a boundary condition that is adapted to electromagnetic optics.

6. The great advantage of (9) as compared to (1) lies in its *rapid convergence for large values of ka*. In order to test this convergence we compute the factors in the denominator of (8) for large ka and ν. If as in (21.30a) we set

$$\cos \alpha = \frac{\nu}{\varrho}, \tag{11}$$

then we have according to (21.39)

$$\zeta_\nu (\varrho) = \frac{i}{\varrho \sqrt{\sin \alpha}} \sin z; \tag{11a}$$

where

(11b) $$z = \varrho\,(\sin\alpha - \alpha\cos\alpha) - \pi/4.$$

The roots of $\zeta_\nu = 0$ are given by

(11c) $\sin z = 0,\quad z = z_m = -m\pi,\quad \cos z_m = (-1)^m,\quad \cos^2 z_m = 1.$

Hence according to (11a) we obtain for $z = z_m$, if we ignore the slowly varying factor $\sin\alpha$,

$$\frac{\partial\zeta_\nu}{\partial\alpha} = \frac{i}{\varrho\sqrt{\sin\alpha}}\left(\cos z\,\frac{dz}{d\alpha}\right)_{z=z_m} = i\,\alpha\sqrt{\sin\alpha}\,\cos z_m,$$

therefore by (11)

(11d) $$\eta_\nu = \frac{\partial\zeta_\nu}{\partial\nu} = \frac{\partial\zeta_\nu}{\partial\alpha}\frac{d\alpha}{d\nu} = \frac{-i\,\alpha}{\varrho\sqrt{\sin\alpha}}\cos z_m.$$

On the other hand, for $z = z_m$ we obtain from (11a,b), if we remember that $\sin z_m = 0$,

(11e) $$\begin{aligned}\zeta'_\nu &= \frac{d\zeta_\nu}{d\varrho} = \frac{\partial\zeta_\nu}{\partial\varrho} + \frac{\partial\zeta_\nu}{\partial\alpha}\frac{d\alpha}{d\varrho} = \frac{i}{\varrho\sqrt{\sin\alpha}}\cos z_m\left(\frac{\partial z}{\partial\varrho} + \frac{\partial z}{\partial\alpha}\frac{d\alpha}{d\varrho}\right)_{z=z_m}\\ &= \frac{i}{\varrho}\sqrt{\sin\alpha}\,\cos z_m.\end{aligned}$$

Finally we obtain from (11d,e) for $\varrho = ka$

(12) $$\eta_\nu(ka)\;\zeta'_\nu(ka) = \alpha\left(\frac{\cos z_m}{ka}\right)^2 = \frac{\alpha}{(ka)^2}.$$

If we substitute this in (8) we obtain

(13) $$D_\nu = -\frac{k}{4\,\alpha}\,\frac{2\nu+1}{\sin\nu\pi}\,\zeta_\nu(kr_0).$$

According to (21.41) ν in the first approximation is equal to ka, but with increasing m it increases in the positive imaginary direction. Hence $\sin\nu\pi$ increases exponentially in the sequence $\nu_1, \nu_2, \ldots$ and hence D_ν decreases exponentially due to the denominator $\sin\nu\pi$.

The same thing occurs in (13) due to the factor $\zeta_\nu(kr_0)$. This latter factor is to be computed according to Debye's formula (21.32) (the higher saddle point is the determining one) and not according to (11a) (where the saddle points are approximately of the same height). Hence the auxiliary angle now has a meaning different from that in (11): $\cos\alpha$ is equal to ν/kr_0 and not to ν/ka as before. Hence, $\zeta_\nu(kr_0)$

also decreases exponentially on the ν- sequence. The same thing holds for the additional factor $\zeta_\nu(k\,r)$ in (9).

In the special case of wireless telegraphy (appendix to Chapter VI) we would need about 1000 terms for a representation of the type (1), whereas, as we shall see, we need only one or two terms of the corresponding series of the type (9). In this appendix we shall also discuss how one type is obtained from the other by a *purely mathematical transformation* (in the complex plane of the index ν).

7. The *structure* of Green's function and its *singular behavior for* $\vartheta = 0$ becomes particularly clear in our representation (9a). In order to obtain a rough estimate for the behavior of Green's function for small ϑ we use the approximation formula (24.32)

$$P_\nu(-\cos\vartheta) \rightarrow \frac{\sin \nu\pi}{\pi} \log \vartheta^2 \quad \text{for} \quad \vartheta \rightarrow 0$$

and then obtain from (9a)

$$G = -\frac{k}{4\pi} \log \vartheta^2 \sum_\nu Z_\nu(k\,r_0)\, Z_\nu(k\,r)\,. \tag{14}$$

Here $\sum_\nu$ has a " δ-like character." Namely, if we expand a function in the fundamental interval $a < r < \infty$ in an arbitrary normalized orthogonal system of functions $Z_\nu\,(k\,r)$:

$$\delta(r, r_0) = \begin{Bmatrix} 0 \ldots\ldots r \neq r_0 \\ \infty \ldots\ r = r_0 \end{Bmatrix} = \sum A_\nu\, Z_\nu(k\,r) \ldots \text{with} \int\limits_{r_0-\varepsilon}^{r_0+\varepsilon} \delta(r, r_0)\, dr = 1\,,$$

then we obtain formally:

$$A_\nu = \int\limits_0^\infty \delta(r, r_0)\, Z_\nu(k\,r)\, dr = Z_\nu(k\,r_0) \int\limits_{r_0-\varepsilon}^{r_0+\varepsilon} \delta(r, r_0)\, dr = Z_\nu(k\,r_0)\,,$$

but as yet we know nothing about the convergence of this general δ- series

$$\delta(r, r_0) = \sum Z_\nu(k\,r_0)\, Z_\nu(k\,r) \qquad \text{for} \quad r \neq r_0 \tag{14a}$$

Only the divergence for $r = r_0$ (all terms positive) is apparent. The "representation of zero" for $r \neq r_0$ is obtained by an infinitely rapid "oscillation around zero." Hence (14a) is not suitable for the actual computation of G for $\vartheta \rightarrow 0$.

We obtain this representation from the defining equation (7a) of the function g, which is continuous throughout, and which for $r = a$ assumes the boundary values

(14b) $$g = g_a = -\frac{e^{ikR_a}}{R_a}, \quad R_a^2 = a^2 + r_0^2 - 2\,a\,r_0 \cos\vartheta_a;$$

here ϑ_a is the polar distance on the sphere $r = a$. In addition, g must satisfy the differential equation $\Delta g + k^2 g = 0$ in the exterior of the sphere $a < r < \infty$, and the radiation condition at infinity. Hence g can be computed as a solution of the exterior boundary value problem on p. 214, which can be represented in terms of Green's function by

$$g = \int g_a \frac{\partial G}{\partial n} d\sigma_a, \qquad \frac{\partial G}{\partial n} = -\left(\frac{\partial G}{\partial r}\right)_{r=a}.$$

Using the representation (9) of G we obtain for g a series summed over ν , which, on the ray $\vartheta = 0$, $a < r < \infty$, reads:

(14c) $$g(r, \vartheta = 0) = -\frac{\pi}{2} \sum \frac{2\nu + 1}{\sin \nu\pi} C_\nu \frac{\zeta_\nu(kr)}{\eta_\nu(ka)},$$

(14d) $$C_\nu = \int_0^\pi g_a P_\nu(-\cos\vartheta_a) \sin\vartheta_a \, d\vartheta_a.$$

Since the singularity $\vartheta = 0$ of $P_\nu(-\cos\vartheta)$ now occurs only under the integral in (14d) and is only of logarithmic order, the coefficients C_ν are all finite; however their explicit computation[17] does not seem to be easy.

In this appendix we have introduced an entirely new kind of "*singular eigenfunctions,*" which are essentially different from the "*regular eigenfunctions*" that we have used so far in this chapter. Our singular eigenfunctions

$$\zeta_\nu(kr)\, P_\nu(-\cos\vartheta)$$

are not *free* oscillations, but require a *stimulation* along the ray $\vartheta = 0$. On the other hand each of the particular solutions

$$u_n(k, r)\, P_n(\cos\vartheta) \quad \text{and} \quad \zeta_n(kr)\, P_n(\cos\vartheta)$$

in (1) is *source free* throughout the physical domain $a \leq r < \infty$, $0 \leq \vartheta \leq \pi$ Their stimulation, if we should speak of one, takes place in the *exterior* of this domain, namely in the center $r = 0$, and for u_n at infinity.

In the author's 1912 paper, quoted on p. 183, our "regular" eigenfunctions were called "*improper*" and our "singular" eigenfunctions were called "*proper.*" The following discussion may serve to justify this apparently paradoxical notation.

[17] See the discussion of "Whittaker's integral," which is a limiting case of our C_ν, in the textbook by Watson, pp. 239–240.

We start from the fact that for all oscillation problems, whether free or forced, periodic or damped, we have the relation

$$(15) \qquad c = \frac{\omega}{k}$$

where we may assume c (the velocity of sound or light) to be real and the carrying medium to be absorption free. Heretofore we have assumed ω to be real and the time dependence to be in the form $\exp\{-i\omega t\}$. The equation

$$(15a) \qquad a \cos\alpha = \frac{\nu}{k}$$

which follows from (11), and our condition $\zeta_\nu(ka) = 0$, then yielded a *complex* value of ν *with positive imaginary part*, so that for real k the quantity $a \cos\alpha$ also had the same character and hence was of the form

$$(15b) \qquad a\cos\alpha = A + iB \text{ with } B > 0.$$

Now however, while preserving the relations (15), (15a), (15b), we set ν equal to a *positive integer*, say $= n$. Then from these relations we obtain *complex* values for k and ω with *negative imaginary parts:*

$$k = k_1 - i k_2 = \frac{n}{A + iB} = n\frac{A - iB}{A^2 + B^2}, \quad \omega = \omega_1 - i\omega_2 = c(k_1 - i k_2).$$

The boundary condition $\zeta_\nu(ka) = 0$ now becomes

$$(15c) \qquad \zeta_n\{(k_1 - i k_2)a\} = 0$$

and the above purely periodic time dependence factor $\exp\{-i\omega t\}$ becomes

$$\exp\{-i\omega_1 t\} e^{-\omega_2 t} = \exp\{-i c k_1 t\} e^{-ck_2 t}.$$

Appending this time factor and considering the regular character of $P_n(-\cos\vartheta)$ for integral n we obtain from our singular eigenfunction (14) the *damped oscillation*

$$(16) \qquad \zeta_n[(k_1 - i k_2)r]\, P_n(-\cos\vartheta) \exp\{-i c k_1 t\} e^{-ck_2 t},$$

which is *regular* throughout the region $r > a$. The amplitude is infinite at $t = -\infty$, decreases at a constant rate, and vanishes at $t = +\infty$, whereas the frequency remains constant. (The only exception is the surface $r = a$, since there we have $\zeta_n(ka) = 0$ throughout.) There exist ∞^2 such oscillations of zonal character. Their parameters are the integer n and the complex roots of the transcendental equation $\zeta_n(ka) = 0$

which are infinite in number. (For tesseral spherical harmonics the number of possible oscillations would increase to ∞^3.) These damped oscillations are obviously the physically simplest particular solutions of our sphere problem for the boundary condition $u = 0$ and hence deserve the name "proper eigenfunctions." Their close connection with our singular eigenfunctions (16) explain how the latter led to the simplest solution of the boundary value problem.

In the following discussion we apply these eigenfunctions, which so far have been developed only for scalar fields, to the case of the electromagnetic-optic field. We shall see in Chapter VI that this can be done without difficulty. We have to keep in mind the following facts:

1. The boundary condition $u = 0$ must be replaced in the electromagnetic case by

$$\frac{\partial}{\partial r}(r\,u) = 0, \tag{17}$$

as will be shown in Chapter VI.

2. In the transcendental equation $\zeta_\nu(ka) = 0$, or, as we wrote in (15c), $\zeta_n(k a) = 0$, we have to replace the function $\zeta_n(\varrho)$ by

$$\xi_n(\varrho) = \frac{\partial}{\partial r}\{r\,\zeta_n(\varrho)\},$$

which, due to $\varrho = k\,r$, is the same as

$$\xi_n(\varrho) = \zeta_n(\varrho) + \varrho\,\zeta_n'(\varrho). \tag{17 a}$$

In the same manner we have to replace the function η in (6a) by

$$\eta_n(\varrho) = \frac{\partial \xi_n(\varrho)}{\partial n}. \tag{17 b}$$

3. While in the scalar case we had *one* field function u, in the electromagnetic case we have *two* such functions u and v. The function v satisfies the same differential equation as u and a similar boundary condition.

Our damped electromagnetic eigen-oscillations have long been known in the literature. In the case of the *sphere* they were investigated by J. J. Thomson[18] in 1884 as the simplest case of the *Hertz oscillator* which was then the center of interest. They were generalized by M. Abraham[19] to the case of the elongated *ellipsoid of revolution* (rod-like oscillator) and the *Paraboloid of revolution* (wire with free ends). Indeed,

[18] *London Math. Soc. Proc.* **15**, 197, and the textbook Recent Researches in Electricity and Magnetism, Oxford 1893, the so-called "third volume of Maxwell."

[19] *Ann. Physik* **66**, 435 (1898); **67**, 834 (1899); *Math. Ann.* **52**, 81 (1899). For further literature see *Enzykl. d. Math. Wiss.* v. V. 2, Abraham's article, p. 508.

our entire development can, with the introduction of elliptic coordinates, be adapted without fundamental change from the cylindrical and spherical harmonics of the representation (16) to the domain of the Lamé wave functions. We can use this method in order to construct Green's function for the exterior of an ellipsoid or paraboloid and thus obtain general solutions to the associated boundary value problems.

We finally indicate some problems for which the method of this appendix is helpful.

a) *Dispersion by colloidal particles.* In a 1908 paper G. Mie deduced the impressive color phenomena seen in the ultramicroscope from the dielectricity constant and the conductivity of the individual scattering particles. The particle was assumed to be spherical with a diameter *small compared to the wavelength,* i.e., $ka \ll 1$. In this case the series of type (1) converge sufficiently rapidly. In the opposite case $ka \gg 1$ the use of geometrical optics suffices; but the intermediate case gives rise to difficulties. In this intermediate case we have to use series of the type (9) as specialized for a source at infinity. The fact that in our case the sphere was assumed to be infinitely conductive, while in Mie's case it was assumed to be an arbitrary dispersive medium, does not make an important difference. We must merely replace the *boundary condition* (equation (17)) for the complete conductor by a *transition condition* between the interior and the exterior. The convergence of the series will be the better the nearer we are to the limiting case of geometrical optics.

b) *The reflection of a plane wave on the surface of a completely conductive sphere.* The diffraction field in the *rear* of a sphere was discussed schematically (i.e., with the simplified boundary condition $u = 0$ and for a scalar field) under 5 above and was represented by the equations (10), (10a) for $ka \gg 1$. On the *front* of the sphere, especially for $\vartheta = 0$, we know from experience of strange interference phenomena, which so far have not been amenable to the usual treatment by series of type (1). The analytical difficulties which arise here are expressed by the singularity of the ray $\vartheta = 0$ in series of the type (9). However, we claim that this problem can be treated in the manner indicated on p. 221, if we take into consideration the actual conditions of the reflection problem.

c) *The rainbow.* With this classic problem we return to the starting point of Debye's asymptotic investigations (see p. 117) and all subsequent advances in the domain of short waves ($ka \gg 1$). The rainbow problem has since been brought to a beautiful conclusion by B. Van der Pol and H. Bremmer.[20] However, from the viewpoint of method there remains a gap between the wave-optical and geometric-optical method.

It was the task of this appendix to bridge such gaps mathematically.

[20] *Phil. Mag.* **24**, 141, 825 (1937).

Appendix III

The Wave Mechanical Eigenfunctions of the Scattering Problem in Parabolic Coordinates

In the following discussion we outline the steps which lead to the representation (30.7). For details the reader is referred to textbooks on wave mechanics.[21]

The parabolic coordinates $\xi = r + x$, $\eta = r - x$ define, in a plane which passes through the x-axis, a double system of confocal parabolas which have the point $r = 0$ as a common focus. The degenerate parabolas $\xi = 0$, $\eta = 0$ coincide with the negative and positive x-axis respectively; the parabolas $\xi = \infty$, $\eta = \infty$ limit the plane in the direction of large positive and negative x respectively.

If we rotate the plane around the x-axis then ξ, η together with the rotation angle φ form a spatial coordinate system which bears the following relation to the Cartesian coordinates x,y,z:

$$x = \frac{1}{2}(\xi - \eta), \qquad y = \sqrt{\xi\,\eta}\cos\varphi, \qquad z = \sqrt{\xi\,\eta}\sin\varphi .$$

From this we obtain the line element

$$ds^2 = \frac{1}{4}(\xi + \eta)\left(\frac{d\xi^2}{\xi} + \frac{d\eta^2}{\eta}\right) + \xi\,\eta\,d\varphi^2 . \tag{1}$$

With its help $\Delta\psi$ is transformed into

$$\Delta\psi = \frac{4}{\xi + \eta}\left(\frac{\partial}{\partial\xi}\,\xi\,\frac{\partial\psi}{\partial\xi} + \frac{\partial}{\partial\eta}\,\eta\,\frac{\partial\psi}{\partial\eta}\right) + \frac{1}{\xi\eta}\frac{\partial^2\psi}{\partial\varphi^2} .$$

The wave equation (29.1) for an interaction energy

$$V = \frac{Z Z' e^2}{r} = \frac{2 Z Z' e^2}{\xi + \eta}$$

and for independence from φ, becomes

$$\frac{\partial}{\partial\xi}\,\xi\,\frac{\partial\psi}{\partial\xi} + \frac{\partial}{\partial\eta}\,\eta\,\frac{\partial\psi}{\partial\eta} + \frac{m_\alpha}{2\hbar^2}[(\xi + \eta)\,W - 2\,Z\,Z'\,e^2)]\,\psi = 0 . \tag{2}$$

This can be separated by setting $\psi = \psi_1(\xi)\,\psi_2(\eta)$; with β as the separation constant we then obtain:

[21] For example, the author's Atombau und Spektrallinien, v. II, Chapter V, §6 and Chapter 11, §9. There, in addition, the asymptotic representation (30.8) is derived with the help of a complex integral representation of L that we cannot discuss here.

$$(3)\qquad \frac{d}{d\xi}\,\xi\,\frac{d\psi_1}{d\xi}+\left(\frac{m_\alpha W}{2\hbar^2}\,\xi-\frac{m_\alpha Z Z' e^2}{2\hbar^2}+\beta\right)\psi_1=0$$

$$(4)\qquad \frac{d}{d\eta}\,\eta\,\frac{d\psi_2}{d\eta}+\left(\frac{m_\alpha W}{2\hbar^2}\,\eta-\frac{m_\alpha Z Z' e^2}{2\hbar^2}-\beta\right)\psi_2=0.$$

The function ψ_1 must satisfy the *radiation condition* (28.2) for $\xi\to\infty$ (large positive x). Written in parabolic coordinates (according to (1) ds_ξ is equal to $\frac{1}{2}\,d\xi$ for large ξ, whence $\partial/\partial r$ in (28.2) becomes $\partial/\partial s_\xi=(d\xi/ds_\xi)\;\partial/\partial\xi\equiv 2\;\partial/\partial\xi$, and we have:

$$(5)\qquad \frac{\xi}{2}\left(2\,\frac{d\psi_1}{d\xi}-i\,k\,\psi_1\right)=0 \text{ with } k=\frac{m_\alpha v}{\hbar},\text{ as in (30.7a).}$$

Hence we set $\psi_1=\exp\left(\frac{i\,k}{2}\,\xi\right)$ and get from (3)

$$(6)\qquad \left(-\frac{k^2}{4}\,\xi+\frac{ik}{2}+\frac{m_\alpha W}{2\hbar^2}\,\xi-\frac{m_\alpha Z Z' e^2}{2\hbar^2}+\beta\right)\psi_1=0.$$

The terms with ξ cancel because of the meaning of k and W in equation (30.1). Therefore (6) is satisfied by choosing

$$(7)\qquad \beta=\frac{m_\alpha Z Z' e^2}{2\hbar^2}-\frac{i\,k}{2}.$$

We see that for this ψ_1 equation (3) is satisfied not only asymptotically but for all ξ .

Due to (7) equation (4) becomes

$$(8)\qquad \frac{d}{d\eta}\,\eta\,\frac{d\psi_2}{d\eta}+\left(\frac{m_\alpha W}{2\hbar^2}\,\eta-\frac{m_\alpha Z Z' e^2}{\hbar^2}+\frac{ik}{2}\right)\psi_2=0.$$

The function ψ_2 must satisfy the *absorption condition* for $\eta\to\infty$ (large negative x), which, written in analogy to (5), reads:

$$\frac{\eta}{2}\left(2\,\frac{d\psi_2}{d\eta}+i\,k\,\psi_2\right)=0.$$

Hence for large η we have the first approximation

$$\psi_2=\exp\left(-\,i\,k\,\eta/2\right).$$

However, this is not an exact solution of (8). Hence we set the more general

$$\psi_2=e^{-ik\eta/2}\,f(\eta).$$

From (8) we obtain the equation for $f(\eta)$

$$(9)\qquad \eta\,\frac{d^2 f}{d\eta^2}+(1-i\,k\,\eta)\,\frac{df}{d\eta}-m_\alpha\,Z\,Z'\,\frac{e^2}{\hbar^2}\,f=0.$$

Equation (9) is the differential equation (29.12) of the Laguerre function $L_\mu(\varrho)$, if we set

$$\varrho = i k \eta, \qquad \mu = -\frac{m_a Z Z' e^2}{i k \hbar^2}.$$

The last value coincides with the imaginary total quantum number n of (30.7b). Hence we have

$$f(\eta) = L_n(i k \eta), \qquad \psi_2 = e^{-ik\eta/2} L_n(i k \eta)$$

and finally

(10) $$\psi = \psi_1(\xi)\,\psi_2(\eta) = e^{ik(\xi-\eta)/2} L_n(i k \eta).$$

Thus we have a quick proof of (30.7.)

Appendix IV

Plane and Spherical Waves in Unlimited Space of an Arbitrary Number of Dimensions

After having treated plane and spherical waves in three-dimensional space and plane and cylinder waves in two-dimensional space, we cannot resist the temptation to adapt these formulas to the many-dimensional case. In this connection we shall encounter remarkable generalizations of the ordinary spherical harmonics, the *Gegenbauer polynomials*, and generalized addition theorems of the *Bessel functions*. A systematic approach to these generalizations is again given by our theorem in §27 about the representation of Green's function in terms of the eigenfunctions for the space in question.

A. Coordinate System and Notations

Let the number of dimensions be $p + 2$ so that $p = 0$ represents two-dimensional, and $p = 1$ represents three-dimensional space. On the one hand we use the Cartesian coordinates $x_1, x_2, \ldots, x_{p+2}$, and on the other hand the polar coordinates $r, \vartheta, \varphi_1, \varphi_2 \ldots \varphi_p$. The connection shall be given by

(1)
$$\begin{aligned}
x_1 &= r\cos\vartheta \\
x_2 &= r\sin\vartheta\cos\varphi_1 \\
x_3 &= r\sin\vartheta\sin\varphi_1\cos\varphi_2 \\
&\text{- - - - - - - - - - -} \\
x_{p+1} &= r\sin\vartheta\sin\varphi_1\sin\varphi_2\ \ldots\ \sin\varphi_{p-1}\cos\varphi_p \\
x_{p+2} &= r\sin\vartheta\sin\varphi_1\sin\varphi_2\ \ldots\ \sin\varphi_{p-1}\sin\varphi_p .
\end{aligned}$$

In order to cover the whole space $-\infty < x_i < +\infty$ the coordinates $r, \vartheta, \ldots, \varphi_p$ must vary between the limits

$$0 < r < \infty, \quad 0 < \vartheta < \pi, \quad \begin{cases} 0 < \varphi_i < +\pi, & i = 1, 2, \ldots, p-1, \\ -\pi < \varphi_p < +\pi. \end{cases}$$

By forming the sum of the squares in (1) we obtain

$$\sum_{i=1}^{p+2} x_i^2 = r^2. \tag{1a}$$

The definition of the $(p+2)$-dimensional line element is

$$ds^2 = \sum_{i=1}^{p+2} dx_i^2. \tag{1b}$$

If for every direction of the coordinates r, ϑ, φ_i we compute the corresponding ds from (1) then we obtain (in unified form)

$$ds = (dr,\ r\,d\vartheta,\ r \sin\vartheta\, d\varphi_1,\ r \sin\vartheta \sin\varphi_1 d\varphi_2, \ldots\ r \sin\vartheta \sin\varphi_1 \ldots \sin\varphi_{p-1} d\varphi_p). \tag{2}$$

The coefficients of $dr, d\vartheta, d\varphi_1, \ldots, d\varphi_p$ on the right side of (2) will be denoted by $g_1, g_2, \ldots, g_{p+2}$. We then have

$$\begin{aligned} g_1 = 1,\ g_2 = r,\ g_3 = r \sin\vartheta,\ g_4 = r \sin\vartheta \sin\varphi_1, \\ \ldots, g_{p+2} = r \sin\vartheta \sin\varphi_1 \ldots \sin\varphi_{p-1}. \end{aligned} \tag{2a}$$

From (2) and (2a) we obtain the $(p+2)$-dimensional volume element

$$d\tau = \bar{g}\, dr\, d\vartheta\, d\varphi_1 \ldots d\varphi_p, \tag{2b}$$

$$\bar{g} = \prod_{i=1}^{p+2} g_i = r^{p+1} \sin^p \vartheta \sin^{p-1} \varphi_1 \sin^{p-2} \varphi_2 \ldots \sin\varphi_{p-1}. \tag{2c}$$

We denote the surface element of the unit sphere in $(p+2)$-dimensional space by $d\omega$, its total surface by Ω and set

$$\Omega = \int d\omega = \Omega_\vartheta\, \Omega_\varphi, \tag{2d}$$

where Ω_ϑ and Ω_φ are the components obtained through integration of $d\omega$ with respect to ϑ and $\varphi_1, \varphi_2, \ldots, \varphi_p$ respectively. From (2b,c) we obtain

$$\Omega_\vartheta = \int_0^\pi \sin^p \vartheta\, d\vartheta = \frac{\pi}{2^p} \frac{\Gamma(p+1)}{\Gamma\left(\frac{p}{2}+1\right) \Gamma\left(\frac{p}{2}+1\right)} = \frac{\pi}{p} \frac{2^{-p+2}\Gamma(p)}{\Gamma(p/2)\, \Gamma(p/2)}. \tag{2e}$$

$$\begin{aligned} \Omega_\varphi &= \int_0^\pi \sin^{p-1} \varphi_1\, d\varphi_1 \int_0^\pi \sin^{p-2} \varphi_2\, d\varphi_2 \ldots \int_0^\pi \sin\varphi_{p-1}\, d\varphi_{p-1} \int_{-\pi}^{+\pi} d\varphi_p \\ &= 2\pi \frac{\frac{p+1}{2}}{\Gamma\left(\frac{p+1}{2}\right)}. \end{aligned} \tag{2f}$$

We denote the Laplace operator in our space by Δ_p (thus in three-dimensional space we would denote it by Δ_1) and
we write for a function u which depends on r alone

(3) $$\Delta_p u = \frac{1}{\bar{g}} \frac{d}{dr}\left(\frac{\bar{g}}{g_1^2} \frac{du}{dr}\right) = \frac{1}{r^{p+1}} \frac{d}{dr}\left(r^{p+1} \frac{du}{dr}\right).$$

The potential equation $\Delta_p u = 0$ then becomes

(3a) $$\frac{d^2 u}{dr^2} + \frac{p+1}{r} \frac{du}{dr} = 0.$$

Except for additive and multiplicative constants of integration we obtain the solution:

(4) $$u = r^{-p}.$$

We generalize this solution to

(4a) $$u = R^{-p}, \qquad R^2 = \sum (x_i - y_i)^2.$$

If we place the second point introduced here on the axis $\vartheta = 0$ and denote its distance from the origin by r_0 then according to (1) we have $y_1 = r_0$, $y_2 = y_3 = \cdots = y_{p+2} = 0$ and $R^2 = r^2 - 2\,r\,r_0 \cos\vartheta + r_0^2$. Hence

(4b) $$u = \frac{r_0^p}{R^p} = \frac{1}{\left[1 + \left(\frac{r}{r_0}\right)^2 - 2\frac{r}{r_0}\cos\vartheta\right]^{p/2}}$$

is also a solution of $\Delta_p u = 0$.

As in (22.3) we expand (4b) in ascending (or descending) powers of r/r_0 and call the coefficients *p-dimensional zonal spherical harmonics*

$$P_n(\cos\vartheta \,|\, p)$$

or also *Gegenbauer polynomials*.[22] The *Legendre polynomials* may thus be denoted by

$$P_n(\cos\vartheta \,|\, 1)\,.$$

Hence we write[23]

(5) $$\left[1 + \left(\frac{r}{r_0}\right)^2 - 2\frac{r}{r_0}\cos\vartheta\right]^{-p/2} = \sum_{n=0}^{\infty} \left(\frac{r}{r_0}\right)^n P_n(\cos\vartheta \,|\, p),$$

and deduce from this

[22] Gegenbauer's original notation (see e.g., *Wien. Akad.* **70** (1875)) is $C_n^{\nu}(\cos\vartheta)$, where $\nu = p/2$.

[23] The defining equation (5) is not limited to integral p; equation (5) breaks down for $p = 0$ since in that case (4b) has to be replaced by the two-dimensional logarithmic potential.

(5a)

$$\begin{array}{l|l}
\cos\vartheta = 1 & P_n(1|p) = (-1)^n\binom{-p}{n} = \dfrac{(p+n-1)!}{n!(p-1)!} \\
\cos\vartheta = -1 & P_n(-1|p) = (-1)^n P_n(1|p) \\
 & P_{2s+1}(0|p) = 0 \\
\cos\vartheta = 0 & P_{2s}(0|p) = \binom{-p/2}{s} = (-1)^s \dfrac{\Gamma\left(\frac{p}{2}+s\right)}{\Gamma(s+1)\Gamma\left(\frac{p}{2}\right)}.
\end{array}$$

For the particular solution of the potential equation $\Delta_p u = 0$ which depends only on r and ϑ

(5b) $$u = r^n P_n(\cos\vartheta|p)$$

we then obtain the differential equation

$$\frac{1}{\bar{g}}\left\{\frac{\partial}{\partial r}\left(\frac{\bar{g}}{g_1^2}\frac{\partial u}{\partial r}\right) + \frac{\partial}{\partial\vartheta}\left(\frac{\bar{g}}{g_2^2}\frac{\partial u}{\partial\vartheta}\right)\right\} = 0\,;$$

and after dividing out the factor $r^{n+p-1}/\bar{g}$ we obtain the ordinary differential equation for $P_n(\cos\vartheta|p)$

(5c) $$\left[\frac{d}{d\vartheta}\sin^p\vartheta\frac{d}{d\vartheta} + n(n+p)\sin^p\vartheta\right] P_n(\cos\vartheta|p) = 0\,.$$

The reader is asked to check the connection of these and the following formulas with the formulas from the theory of ordinary spherical harmonics.

The Gegenbauer polynomials can be expressed in terms of hypergeometric series in a manner similar to that of the Legendre polynomials in (24.24a); we have

(5d) $$P_n(\cos\vartheta|p) = P_n(1|p)\, F\left(-n, n+p, \frac{p+1}{2}, \frac{1-\cos\vartheta}{2}\right).$$

B. The Eigenfunctions of Unlimited Many-Dimensional Space

From the potential equation we pass to the wave equation. For a function which depends only on r the wave equation is, according to (3),

(6) $$\frac{d^2u}{dr^2} + \frac{p+1}{r}\frac{du}{dr} + k^2 u = 0.$$

If we set $u = r^{-p/2} w$, then we obtain the Bessel differential equation with index $p/2$ for w. Hence (6) is integrated by

(6a) $$u = r^{-p/2} I_{p/2}(kr),$$

and also by

(6b) $$u = r^{-p/2} H^1_{p/2}(kr),$$ (6c) $$u = r^{-p/2} H^2_{p/2}(kr).$$

The function in (6b) behaves asymptotically like

$$C e^{ikr} \Big/ r^{\frac{p+1}{2}}, \qquad C = \sqrt{\frac{2}{k\pi}}\, e^{-\frac{p+1}{2}\frac{i\pi}{2}}$$

and satisfies the radiation condition (28.7)

$$\lim_{r\to\infty} r^{\frac{p+1}{2}} \left(\frac{\partial u}{\partial r} - i k u\right) = 0;$$

In the same manner (6c) satisfies the absorption condition. Hence (6b,c) represent the radiated and absorbed *spherical waves in* $(p+2)$-*dimensional space*. This remains valid for a general position of the source point with the solutions

(7) $$U = R^{-p/2} H_{p/2}(kR), \quad R^2 = r^2 - 2 r r_0 \cos\vartheta + r_0^2.$$

The function in (6a) may be called "eigenfunction of *spherical* symmetry." We now want to find the general eigenfunctions of zonal symmetry. They are of the form

(8) $$u_n(r,\vartheta) = v_n(r) P_n(\cos\vartheta|p).$$

From the equation (5c) of P_n we find the differential equation of v_n

$$\left(\frac{d^2}{dr^2} + \frac{p+1}{r}\frac{d}{dr} + k^2 - \frac{n(n+p)}{r^2}\right) v_n = 0.$$

If we treat this equation as we did (6) by setting $v_n = r^{-p/2} w$ then for w we obtain the Bessel differential equation with index $n + p/2$, and hence as the solution which is finite for $r = 0$

$$w = I_{n+p/2}(kr).$$

Hence the eigenfunction becomes

(8a) $$u_n = r^{-p/2} I_{n+p/2}(kr) P_n(\cos\vartheta|p).$$

According to §26 any two of these eigenfunctions are mutually *orthogonal*, both in the continuous spectrum $0 < k < \infty$, and in the discrete spectrum $n = 0,1,2, \ldots$.

For two eigenfunction u_n, u_m with equal k but different indices we obtain from (2b,c) and (8):

(9) $$\int u_n u_m d\tau = \int_0^\infty I_{n+p/2}(kr) I_{m+p/2}(kr)\, r\, dr \int_0^\pi P_n(\cos\vartheta|p) P_m(\cos\vartheta|p) \sin^p\vartheta\, d\vartheta\, \Omega_\varphi.$$

where Ω_φ is as in (2f). Due to the fact that neither Ω_φ nor the integral with respect to r vanish and due to the orthogonality of u_n and u_m we obtain:

(10) $$\int_0^\pi P_n(\cos\vartheta\,|\,p) P_m(\cos\vartheta\,|\,p) \sin^p\vartheta\, d\vartheta = 0, \quad m \neq n.$$

Note the characteristic factor $\sin^p\vartheta$ in (10), which in the three-dimensional case ($p = 1$) becomes the customary factor $\sin\vartheta$ for the Legendre polynomials. While in the customary analytic derivation of (10) this factor might appear artificial, it follows in our many-dimensional approach directly from the meaning of $d\tau$.

We also note the corresponding normalizing integral for $m = n$

$$(11)\qquad N = \int_0^\pi [P_n(\cos\vartheta \mid p)]^2 \sin^p\vartheta\, d\vartheta = \frac{\Gamma(n+p)}{2^{p-1}(n+p/2)\, n!}\, \frac{\pi}{\Gamma(p/2)\,\Gamma(p/2)}\,.$$

which is a generalization of the normalizing integral for ordinary zonal spherical harmonics: $N = 1/(n+\frac{1}{2})$ for $p = 1$. The proof of (11) starts from the defining equation (5) of the Gegenbauer polynomials.

With the help of (2e) we can replace (11) by:

$$(11\text{a})\qquad N = \frac{p}{2}\, \frac{\Gamma(n+p)}{(n+p/2)\, n!\, \Gamma(p)}\, \Omega_\vartheta .$$

C. Spherical Waves and Green's Function in Many-Dimensional Space

The spherical wave of zonal symmetry has been described by equation (7). From this function we obtain Green's function of $(p+2)$-dimensional unlimited space by adding a factor f such that the source Q of U becomes a unit source. According to §10 C this means

$$(12)\qquad 1 = \int \frac{\partial G}{\partial n}\, d\sigma = f \int \frac{\partial U}{\partial R}\, d\sigma = f \int \frac{\partial}{\partial R}\, [R^{-p/2} H^1_{p/2}(k R)]\, R^{p+1}\, d\omega .$$

where the integration is to be taken over a sphere of radius $R \to 0$; $d\sigma$ denotes the surface element on this sphere; $d\omega$, as in (2d), denotes the surface element on the unit sphere. Hence we obtain from (12)

$$(12\text{a})\qquad 1 = f\,\Omega \lim_{R\to 0} R^{p+1} \frac{\partial}{\partial R}\, [R^{-p/2} H^1_{p/2}(k R)].$$

For odd p we can use the formula (19.31) for H, which yields

$$\lim_{R\to 0} \cdots = \frac{-i R^{p+1}}{\sin p\pi/2}\, \frac{\partial}{\partial R}\, [R^{-p/2} I_{-p/2}(k R)] = \frac{i p}{\sin p\pi/2}\, \frac{(k/2)^{-p/2}}{\Gamma(-p/2+1)}\,.$$

Using a well-known Γ- relation we can replace this by

$$\frac{i p}{\pi} \left(\frac{k}{2}\right)^{-p/2} \Gamma\left(\frac{p}{2}\right).$$

For even p we obtain the same value from (19.26) and (19.47). Hence we obtain from (12a)

(12 b) $$f = \frac{\pi}{\Omega}\left(\frac{k}{2}\right)^{p/2} \Big/ i\,p\,\Gamma\left(\frac{p}{2}\right)$$

and from (7) upon multiplication by f

(13) $$G(P,Q) = \frac{\pi}{\Omega}\left(\frac{k}{2}\right)^{p/2}\left(\frac{kR}{2}\right)^{-p/2} H^1_{p/2}(kR) \Big/ i\,p\,\Gamma\left(\frac{p}{2}\right).$$

On the other hand we want to construct $G(P,Q)$ as in §28 from the eigenfunctions $u(P)$ in (8a)

(13 a) $$u(P) = r^{-p/2} I_{n+p/2}(\lambda r)\, P_n(\cos\vartheta|p)$$

and the associated $u(Q)$ for a point Q with the coordinates $r = r_0$, $\vartheta_0 = 0$:

(13 b) $$u(Q) = r_0^{-p/2} I_{n+p/2}(\lambda r_0)\, P_n(1|p).$$

In both representations (13a,b) λ (see equation (28.14)) denotes the variable of integration in the continuous part of the eigen value spectrum. In a similar manner we perform the integration over λ in the complex λ- plane and obtain in analogy to (28.15)

(14) $$\int u(P)\,u(Q)\,\frac{d\lambda}{k^2-\lambda^2}$$
$$= \frac{\pi i}{2}(r r_0)^{-p/2}\sum_n P_n(\cos\vartheta|p)\,P_n(1|p)\begin{cases} I_{n+p/2}(kr_0)\,H_{n+p/2}(kr) & r > r_0, \\ I_{n+p/2}(kr)\,H_{n+p/2}(kr_0) & r < r_0. \end{cases}$$

In order to be able to apply this formula to Green's function we still must normalize the functions $u(P)$ and $u(Q)$ to one. The general term on the right side of (14) must therefore be: 1) divided by the normalizing factor N of (11a) due to the dependence on ϑ; 2) divided by Ω_φ of (2f) due to its independence of the coordinates $\varphi_1, \varphi_2, \ldots, \varphi_p$, and 3) multiplied by k due to the r-dependence according to Appendix I, equation (4). Altogether this yields the factor (see also (5a))

(14 a) $$\frac{k}{\Omega}\,\frac{2n+p}{p}\,\frac{n!\,\Gamma(p)}{\Gamma(n+p)} = \frac{k}{\Omega}\,\frac{2n+p}{p} \Big/ P_n(1|p),$$

which has to be introduced under the Σ- sign of (14). Thus, according to our general theorem of §28 we obtain Green's function of unlimited space. Comparing this with (13) we obtain:

(15) $$\frac{H_{p/2}(kR)}{(kR)^{p/2}} = 2^{p/2}\,\Gamma\left(\frac{p}{2}\right)\sum_{n=0}^{\infty}\left(n+\frac{p}{2}\right) P_n(\cos\vartheta|p)\,\Big\{\ \Big\},$$

$$\Big\{\ \Big\} = \begin{cases} \dfrac{I_{n+p/2}(kr_0)}{(kr_0)^{p/2}}\,\dfrac{H_{n+p/2}(kr)}{(kr)^{p/2}} & \cdots\ r > r_0, \\[2ex] \dfrac{I_{n+p/2}(kr)}{(kr)^{p/2}}\,\dfrac{H_{n+p/2}(kr_0)}{(kr_0)^{p/2}} & \cdots\ r < r_0. \end{cases}$$

This *general addition theorem of Bessel functions* holds both for $H = H^1$ and for $H = H^2$, and hence for any linear combination of the two, so that in (15) we may replace H on both sides by

$$Z = c_1 H^1 + c_2 H^2,$$

hence, in particular by

$$I = \tfrac{1}{2}(H^1 + H^2),$$

in which latter case the distinction $r \gtrless r_0$ becomes immaterial. The theorem holds, under more general conditions than those assumed in the derivation, if, as in footnote 22, we replace $p/2$ by an arbitrary number, say ν.

D. Passage from the Spherical Wave to the Plane Wave

For $r_0 \to \infty$ we deduce a representation of the plane wave in many-dimensional space from the last line of (15)

First we obtain on the right side, according to Hankel's approximation (19.55),

$$H^1_{n+p/2}(k r_0) = a\, e^{-i n \pi/2}, \qquad a = \sqrt{\frac{2}{\pi k r_0}} \exp\left\{ i \left(k r_0 - \frac{p+1}{2} \frac{\pi}{2} \right) \right\}.$$

Correspondingly, on the left side we obtain

$$H^1_{p/2}(k R) = \sqrt{\frac{2}{\pi k R}} \exp\left\{ i \left(k R - \frac{p+1}{2} \frac{\pi}{2} \right) \right\}.$$

However for $r_0 \to \infty$ we have

$$R = r_0 \left(1 - 2 \frac{r}{r_0} \cos\vartheta + \cdots \right)^{\frac{1}{2}} = r_0 - r\cos\vartheta + \cdots;$$

therefore,

$$H^1_{p/2}(k R) = \sqrt{\frac{2}{\pi k r_0}} \exp\left\{ i \left(k r_0 - k r \cos\vartheta - \frac{p+1}{2} \frac{\pi}{2} \right) \right\}$$
$$= a \exp\{ - i k r \cos\vartheta \},$$

so that the left side of (15), with the corresponding approximation for the denominator, becomes:

$$\frac{a}{(k r_0)^{p/2}} \exp\{ - i k r \cos\vartheta \}.$$

After canceling the common factor on both sides we obtain

$$e^{-ikr\cos\vartheta} = 2^{p/2}\,\Gamma\left(\frac{p}{2}\right)\sum_{n=0}^{\infty}\left(n+\frac{p}{2}\right)e^{-in\pi/2}\,P_n(\cos\vartheta|p)\,\frac{I_{n+p/2}(kr)}{(kr)^{p/2}}. \tag{16}$$

This represents an incoming wave in the direction of the positive axis $\vartheta = 0$, or, in other words, a wave which proceeds in the negative direction of this axis. The wave which proceeds in the positive direction is obtained from (16) by replacing $+\,i$ with $-\,i$. The reader is asked to verify that this formula coincides for $p = 1$ with the three-dimensional representation (24.7). In the two-dimensional case in which (4b) breaks down (see footnote 22) equation (16) is replaced by the representation (21.2b).

Through a suitable averaging or with the help of an "addition theorem[24] of Gegenbauer polynomials" we obtain from (16) remarkable relations between Bessel functions of integral and of fractional indices.[25]

[24] See the lucid collection of Gegenbauer's results in the book by Magnus and Oberhettinger, Formeln und Sätze über die speziellen Funktionen der Mathematischen Physik, Springer, 1943, particularly p. 77.

[25] G. Bauer, *Sitzungsber. bayr. Akad.*, 1875, p. 247; generalization to higher dimensions, A. Sommerfeld, *Math. Ann.* 119, 1 (1943).

CHAPTER VI

Problems of Radio

The problems of signals with electric waves have been in the foreground of applied physics since the beginning of the century. Can we understand the remarkable range of radio signals from the otherwise completely reliable Maxwell theory? The answer is both yes and no. Yes, in so far as only the known electrodynamic laws are applied. No, in so far as the ionosphere (Kenelly-Heaviside layer) plays an essential role in overcoming the curvature of the earth, and has to be added to the Maxwell wave propagation as a *deus ex machina.*

Unfortunately we shall be unable to treat the reflection processes in the ionosphere, and shall restrict ourselves to *questions of propagation in the homogeneous atmosphere and in the earth which is also assumed to be homogeneous.* We shall also have to omit the questions of the construction of transmitters and receivers, which are of such great importance for the engineer, since they do not properly belong to the domain of partial differential equations. Instead, we shall idealize the transmitter to the utmost and treat it as a *Hertz dipole* (§31). On the other hand the questions of propagation definitely belong to our domain and they will give us a complete demonstration of the usefulness of the methods which we have developed above and which we have so far applied mainly to rather artificial problems of heat conduction and of potential theory. Further demonstrations of this usefulness are given by problems in general electrodynamic oscillations. They are treated with some completeness in the textbook by Frank-Mises, Chapter XXIII, and the reader is referred to that book. If, among these problems, we again consider radio, it is because the previous representation was simplified so drastically that it could not be reconciled with practical problems. Now we shall not place our antenna-dipole on the surface of the earth, but at some distance from it, we shall treat the radiation of the horizontal antenna in more detail and demonstrate its asymptotic identity with the radiation of the vertical antenna for increasing distance from the origin, and we shall treat the radiation characteristic with respect to the terms of second order in $1/r$, etc. The energy conditions (required energy supply for prescribed antenna current, heat loss in the earth) will be discussed in the final section. We shall almost always consider the earth as a *plane.* The analytically interesting problem of the earth's curvature, which opens a further domain of application to the method of eigenfunctions, can be treated only in an appendix, since even an only moderately complete treatment of the problem of a plane earth is almost too long for us here.

§ 31. The Hertz Dipole in a Homogeneous Medium Over a Completely Conductive Earth

We assume that the reader has a knowledge of the concepts of electrodynamics and their interconnection through Maxwell's equations. Since we are not dealing with atomic physics but only with the phenomenological Maxwell theory, we shall use the system of the four units, M (meter), K (kilogram mass), S (second), Q (charge, measured in Coulombs). In this system the specific inductive capacity and the permeability are definite quantities; as usual their values in a vacuum are denoted by ε_0 and μ_0. We then have $\varepsilon_0 \mu_0 = 1/c^2$. The parasite factor 4π, which mars the customary electromagnetic equations, is suppressed in our system through the suitable choice of units, wherever it is not implied by the spherical symmetry of the problem.

A. Introduction of the Hertz Dipole

In the electrostatic case we deduce the potential of the dipole by an oriented differentiation from the fundamental potential $\Phi = 1/r$ (see §24 C); the field $\mathbf{E}$ of the dipole is then obtained from this potential by another differentiation. In the electrodynamic case Φ is replaced by the function of the spherical wave

$$(1) \qquad \Pi = \frac{1}{r} e^{ikr}, \quad \text{or more completely} \quad \Pi = \frac{1}{r} e^{i(kr - \omega t)}.$$

The notation Π is due to Hertz[1] himself. As shown by the second form of equation (1), we assume the oscillation to be *purely periodic* and *undamped in time* (this is realized for the tube transmitter).

In the abbreviated first form of (1), which we shall use in the following discussion, we have to remember that

$$(2) \qquad \Pi = -i\omega\Pi = -ikc\Pi.$$

where

[1] In his fundamental work "Die Kräfte elektrischer Schwingungen," collected works II, p. 147, which also contains the well-known force lines of the oscillating dipole.

$$\omega = \quad \text{circular frequency}$$

(2a) $$k = 2\pi/\lambda = \omega/c = \quad \text{wave number}$$

$$c = \omega/k = \quad \text{velocity of light in a vacuum.}$$

As we know, Π satisfies the oscillation equation (7.4), which for purely periodic processes becomes the wave equation:

(3) $$\Delta\Pi + k^2\Pi = 0.$$

In the electrodynamic case Π is not a *scalar* but a *vector*. Hence in the future we shall speak of the *Hertz vector* $\vec{\Pi}$. It is connected with the vector potential **A** by the simple relation

(3a) $$\vec{\Pi} = \mathbf{A}.$$

Just as the individual elements of which **A** is composed have the direction of the corresponding elements of current, so our $\vec{\Pi}$ in empty space (i.e., in the absence of the earth) for a single antenna would have the direction of the antenna current. Here we assume the antenna to be *short compared to the wavelength*, that is, with both ends loaded with capacities so that the current can be considered in the same phase along the whole antenna. In representation (1) we could express the vector character of Π by multiplying the right side of (1) by a constant vector which has the direction of the antenna and, as we shall show later, the dimension of an electric momentum (charge × length). However, we shall refrain from doing this in order not to make the formulas unnecessarily cumbersome; hence we retain equation (1), although it is inconsistent from a vectorial and even a dimensional point of view. Only in §36 shall we correct this flaw. However, we wish to stress now that, due to the vector character of $\vec{\Pi}$, we have to give the Laplace operator Δ in (3) its general vector-analytic meaning

(3 b) $$\Delta\vec{\Pi} = \text{grad div}\,\vec{\Pi} - \text{curl curl}\,\vec{\Pi}.$$

(see v.II, equation (3.10a)). This will be used in §32. Only in this and the following section, where we deal with one Cartesian component Π_z or Π_x at a time, can we use the ordinary Δ.

We now claim that the field **E**, **H** can be obtained from $\vec{\Pi}$ by the following differentiation process:

(4) $$\mathbf{E} = k^2\vec{\Pi} + \text{grad div}\,\vec{\Pi}, \qquad \mathbf{H} = \frac{k^2}{\mu_0 i\omega}\,\text{curl}\,\vec{\Pi}.$$

In order to prove this we must show that Maxwell's equations in a vacuum

$$\mu_0 \dot{\mathbf{H}} + \operatorname{curl} \mathbf{E} = 0\,, \qquad \varepsilon_0 \dot{\mathbf{E}} - \operatorname{curl} \mathbf{H} = 0 \tag{5}$$

are satisfied, where as in (2) we have to replace

$$\dot{\mathbf{H}} \text{ by } -i\omega \mathbf{H}, \qquad \dot{\mathbf{E}} \text{ by } -i\omega \quad \mathbf{E} \tag{5a}$$

Due to (4) and (5a) the left sides of (5) become

$$\operatorname{curl}\ (-k^2 + k^2 + \operatorname{grad}\operatorname{div})\, \vec{\Pi}$$

and

$$-i\omega\,\varepsilon_0 (k^2 + \operatorname{grad}\operatorname{div} - \operatorname{curl}\operatorname{curl})\, \vec{\Pi}.$$

Both vanish, the first due to curl grad $= 0$, the second due to (3) and (3b). Hence, if for $\vec{\Pi}$ we substitute (1) and determine the free constant in terms of the strength of the alternating current in the antenna, then, according to Maxwell, we have in (4) the field radiated from the antenna, valid for all distances that are large compared to $\lambda = 2\pi/k$ For the immediate neighborhood of the antenna our description breaks down owing to the excessive idealization of our antenna model. Following Hertz, we call our model an oscillating or pulsating *dipole*, since the ends of the antenna (both in this picture and in reality) carry alternating opposite charges. This extreme simplification of the antenna, which in reality is of complicated construction, may serve as an example of the degree to which physical data can be idealized in order to make them accessible to fruitful mathematical treatment.

We now pass from the case of vacuum to that of a *medium "earth" of general electromagnetic behavior:* it is still homogeneous but with arbitrary dielectric constant ε and conductivity σ; also its permeability μ will be arbitrary for the time being. The equations (1) and (3) for Π remain formally valid; however the wave number k is no longer determined by (2a) but by

$$k^2 = \varepsilon\mu\omega^2 + i\mu\sigma\omega. \tag{6}$$

At the same time (4) is replaced by:

$$\mathbf{E} = k^2\, \vec{\Pi} + \operatorname{grad}\operatorname{div} \vec{\Pi}, \qquad \mathbf{H} = \frac{k^2}{\mu i \omega} \operatorname{curl} \vec{\Pi}. \tag{7}$$

As before, we prove that the corresponding generalized Maxwell equations

$$\begin{aligned} \mu \dot{\mathbf{H}} + \operatorname{curl} \mathbf{E} &= 0, \\ \varepsilon \dot{\mathbf{E}} + \sigma \mathbf{E} - \operatorname{curl} \mathbf{H} &= 0 \end{aligned} \tag{7a}$$

are satisfied. The oscillation equation, from which we obtained the wave equation by the elimination of time dependence, is obtained in analogy to (7.4):

$$\Delta \Pi = \left(\varepsilon\mu \frac{\partial^2}{\partial t^2} + \sigma\mu \frac{\partial}{\partial t} \right) \Pi . \tag{7 b}$$

B. Integral Representation of the Primary Stimulation

We first wish to bring the representation (1) of Π into the form of a *superposition of eigenfunctions.* Since we are dealing with cylindrical polar coordinates r, φ, z, we shall use the eigenfunctions u and eigen values λ of (26.3) and (26.3a) that are independent of φ; we denote the quantity $m\pi/h$ by μ.[2]
We then have:

$$u = I_0(\lambda r) \cos \mu z, \qquad k^2 = \lambda^2 + \mu^2 . \tag{8}$$

However, whereas the λ has previously been restricted to a discrete spectrum corresponding to the boundary conditions on the cylinder of finite radius, we now have a continuous spectrum $0 \leq \lambda < \infty$ corresponding to the unlimited medium (see §28). Thus, according to (8) the μ also have continuous, and in general, complex values. Furthermore, since we no longer have the boundary condition for the bases of the cylinder, we shall replace $\cos \mu z$ by $\exp(\pm \mu z)$. Hence we are looking for a representation of Π of the form

$$\Pi = \int_0^\infty F(\lambda) I_0(\lambda r) e^{\pm \mu z} d\lambda , \qquad \mu = \sqrt{\lambda^2 - k^2}, \tag{9}$$

where $F(\lambda) d\lambda$ represents the arbitrary amplitude constant by which any eigenfunction may be multiplied. Due to the altered meaning of r (cylindrical coordinate r instead of the spherical polar coordinate r in (1)) we have to rewrite the expression (1) for Π as

$$\Pi = \frac{e^{ikR}}{R}, \qquad R^2 = r^2 + z^2 . \tag{10}$$

Our condition (9) then reads for $z = 0$:

[2] A confusion between this μ and the above magnetic constant μ is unlikely. The latter, moreover, will soon disappear from our formulas.

$$\frac{e^{ikr}}{r} = \int_0^\infty F(\lambda)\, I_0(\lambda r)\, d\lambda . \tag{11}$$

In order to satisfy this condition we use the integral representation of an arbitrary function by the Bessel functions of §21 B. We employ equation (8a) of that section, which for

$$f(r) = \frac{e^{ikr}}{r}, \qquad n = 0$$

becomes

$$\frac{e^{ikr}}{r} = \int_0^\infty \sigma\, \varphi(\sigma)\, I_0(\sigma r)\, d\sigma ,$$

$$\varphi(\sigma) = \int_0^\infty e^{ik\varrho} I_0(\sigma \varrho)\, d\varrho .$$

The first of these equations becomes identical with (11), if we make the following changes in notation

$$\sigma = \lambda, \qquad \sigma\, \varphi(\sigma) = F(\lambda), \quad \text{hence} \quad \varphi(\sigma) = F(\lambda)/\lambda ;$$

The second equation then becomes

$$F(\lambda) = \lambda \int_0^\infty e^{ik\varrho} I_0(\lambda \varrho)\, d\varrho , \tag{11b}$$

which is the *solution of the integral equation* (11). The integration in (11b) can be performed in an elementary fashion, if we use the representation (19.14) for I_0 with the limits of integration $\pm\pi$; namely, by reversing the order of integration we obtain

$$F(\lambda) = \frac{\lambda}{2\pi} \int_{-\pi}^{+\pi} dw \int_0^\infty e^{i\varrho(k+\lambda\cos w)}\, d\varrho = -\frac{\lambda}{2\pi i} \int_{-\pi}^{+\pi} \frac{dw}{k+\lambda\cos w} . \tag{12}$$

The last expression arises from the lower limit $\varrho = 0$ in the preceding integration with respect to ϱ ; the term arising from the upper limit $\varrho = \infty$ can be made to vanish by a small deformation of the path of integration into the "shaded" region of the w-plane (see Fig. 18). The remaining integration with respect to w yields

$$\frac{2\pi}{\sqrt{k^2-\lambda^2}} . \tag{12a}$$

Hence (12) becomes:

$$F(\lambda) = \frac{\lambda}{\sqrt{\lambda^2 - k^2}} = \frac{\lambda}{\mu} \tag{13}$$

and (11) becomes:

$$\frac{e^{ikr}}{r} = \int_0^\infty I_0(\lambda r) \frac{\lambda\, d\lambda}{\mu}. \tag{13a}$$

From (10) we now obtain a corresponding representation for Π Namely, we can complete (13a) to a function of r and z, which satisfies the differential equation (1) by setting:

$$\Pi = \frac{e^{ikR}}{R} = \int_0^\infty I_0(\lambda r)\, e^{-\mu|z|} \frac{\lambda\, d\lambda}{\mu} \tag{14}$$

where $\mu = \sqrt{\lambda^2 - k^2}$ is to be taken with *positive real part,* in order to insure the convergence of the integral and its vanishing in the limit $z \to \pm\infty$. The fact that (14) coincides with (13a) for $z = 0$ insures that it also gives the correct representation of e^{ikR}/R for $z \neq 0$.

In the following section we shall transform (14) into

$$\Pi = \frac{1}{2} \int_{-\infty}^{+\infty} H_0^1(\lambda r)\, e^{-\mu|z|} \frac{\lambda\, d\lambda}{\mu} \tag{14a}$$

with a more exact determination of the path of integration, which will then be complex. Due to the asymptotic character of H_0^1, equation (14a) has the advantage over (14) in that it demonstrates that the radiation condition is satisfied, just as in (1) where the factor $\exp(+ikr)$ is adapted to the radiation condition.

C. Vertical and Horizontal Antenna for Infinitely Conductive Earth

Up to now we have dealt only with unlimited space, whether empty or filled by a homogeneous medium with the constants ε, μ, σ . We now pass to the case of the half-space $z > 0$, which, at $z = 0$, is bounded by an *infinitely conductive earth* ($\sigma \to \infty$) , in which $\mathbf{E} = 0$. Hence, due to the equality of the tangential field strength, which is required by the Maxwell theory, we know that $\mathbf{E}_{\text{tang}}$ must vanish also on the positive side of $z = 0$. According to (7) this means

$$(k^2 \vec{\Pi} + \operatorname{grad} \operatorname{div} \vec{\Pi})_{\text{tang}} = 0 \qquad \text{for } z = 0. \tag{15}$$

We satisfy this condition by adjoining the mirror images of opposite sign to the two single poles of the given dipole: Figure 27a,b serves to illustrate this.

a) Vertical antenna at a distance h above z = 0. The arrows leading from the negative to the positive charge are in the *same direction* for the original dipole and for its mirror image. Hence we write:

$$(16)\qquad \Pi = \Pi_z = \frac{e^{ikR}}{R} + \frac{e^{ikR'}}{R'}, \quad \begin{cases} R^2 = r^2 + (z-h)^2, \\ R'^2 = r^2 + (z+h)^2. \end{cases}$$

The parallelogram on the left side of the drawing shows that charges of the two dipoles equidistant from $z = 0$ act on a hypothetical unit charge situated in the plane $z = 0$ so that the resulting force is in the z-direction. This means $\mathbf{E}_{\text{tang}} = 0$.

b) Horizontal antenna at a distance h above z = 0. The arrow of the reflected dipole has the *opposite* direction to that of the original dipole. Hence we write

$$(17)\qquad \Pi = \Pi_x = \frac{e^{ikR}}{R} - \frac{e^{ikR'}}{R'}.$$

where R and R' are as before.[3] The parallelogram on the right side of the drawing shows that two associated charges of the two dipoles act on a positive unit charge in the plane $z = 0$

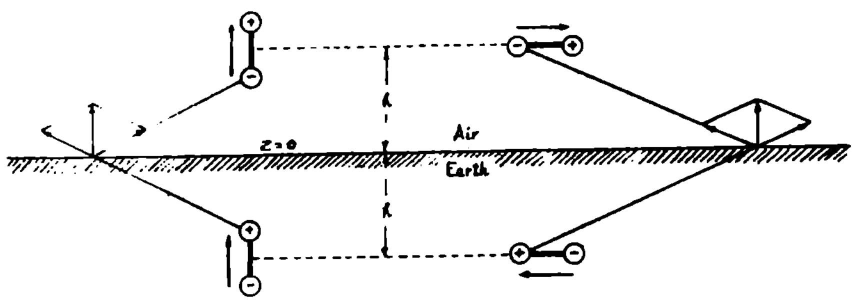


Fig. 27. Reflection by infinitely conductive earth. a) The vertical dipole. The auxiliary construction on the left shows that the horizontal components of the forces exerted by a pair of mirror image poles on a particle on the boundary plane cancel. b) The same thing is shown by the auxiliary construction on the right for the horizontal dipole. For the latter the orientations of the arrows in the original and its mirror image are *opposite*. for the vertical dipole they are *equal*.

so that the resulting force is perpendicular to the plane $z = 0$. Hence we again have $\mathbf{E}_{\text{tang}} = 0$.

[3] The opposite choice of signs in (16) and (17) indicates the vector character of Π, which is suppressed in equation (1).

In exercise VI.1 we shall compute this using the vector formula (15) for both cases a) and b).

In one respect a) and b) differ fundamentally. Namely, if we pass to the limit $h \to 0$, we obtain

$$\Pi = 2\,\frac{e^{ikR}}{R}\,,\ \text{from (16) but}\qquad \Pi \equiv 0\quad\text{from (17)}$$

Hence: a *vertical antenna located directly on the earth* for a sufficient conductivity of the soil generates the field which would be generated by the same antenna in empty space in complete absence of the earth. On the other hand a *horizontal antenna located directly on the earth* for a complete conductivity of the soil is canceled by its mirror image. *The former* made it possible to adapt the formulas and figures of Hertz' original work, which were relative to empty space, to the case of a grounded antenna (Max Abraham). In fact, we can cut the Hertz pattern of force lines of the oscillating dipole along its central plane and replace that plane by the surface of the earth. The force lines are then perpendicular to this plane and hence satisfy condition (15). *The latter*, that is, the disappearance of the horizontal antenna field for $h = 0$ as expressed by (17), decreases rapidly in importance for $h > 0$ (see the figures in §36). Indeed, the horizontal antenna is an effective means of communication even when $h < \lambda$ and the medium is sea water (a very good conductor for the comparatively long radio waves). Thus we see that for the horizontal antenna the nature of the ground and the distance from the ground play a greater role than for the vertical antenna. The formula $\vec{\Pi} = \Pi_x$ in (17) is then no longer adequate and must be generalized (see §33).

D. Symmetry Character of the Fields of Electric and Magnetic Antennas

As we have just seen, the vertical antenna gives the field of a Hertz *dipole* of strength 2 for the limit $h \to 0$, and the horizontal antenna yields a zero field. However, if in the latter case we let the antenna current increase at the rate at which h decreases, then we obtain the field of a *quadrupole*. In fact under this limit process Fig. 27b goes over into the quadrupole scheme as seen on p. 152. Replacing the amplitude factors 2 and zero by A and B we can write:

$$\text{(18)}\quad \text{Vertical antenna:}\qquad \Pi_z = A\,\frac{e^{ikR}}{R}\qquad \text{Dipole,}$$

$$\text{Horizontal antenna:}\quad \Pi_x = B\,\frac{\partial}{\partial x}\,\frac{e^{ikR}}{R}\qquad \text{Quadrupole.}$$

The latter representation corresponds in Fig. 27b to the combination of the pairs of poles which lie on the same vertical line to a vertical dipole and to their relative translation in the horizontal direction. This means that the horizontal antenna in the x-direction is equivalent to vertical antennas with opposite current that are mutually translated in the x-direction. We shall discuss this more closely in connection with Fig. 30. Written in polar coordinates $x = r\cos\varphi$, $y = r\sin\varphi$ the second formula (18) reads

$$\Pi_x = B\frac{x}{r}\frac{\partial}{\partial r}\frac{e^{ikR}}{R} = B\cos\varphi\,\frac{r}{R}\frac{d}{dR}\frac{e^{ikR}}{R}\,. \tag{18a}$$

Hence, the directions $\varphi = 0$ and $\varphi = \pi$ parallel to the antenna are preferred directions for Π_x; in the perpendicular directions $\varphi = \pm\pi/2$ Π_x vanishes. The associated direction characteristics of the horizontal antenna will be described in Fig. 29; where we shall also compute the constant B (which vanishes with increasing conductivity). On the other hand the field of the vertical antenna is symmetric with respect to the z-axis and hence its direction characteristic is a circle. From this follows the particular suitability of the horizontal antenna for directed broadcasts (see §33).

Rod antennas of vertical or horizontal direction are called *electric transmitters.* A coil traversed by an alternating current or any (circular, rectangular, etc.) closed conductor is called a *magnetic transmitter,* because then the magnetic field is concentrated in the axis of the coil (the normal of the wire loop); the customary notation is "frame antenna." In the central perpendicular of the frame a magnetic alternating current pulsates, while along the rod antenna there pulsates an electric alternating current. While the *magnetic* force lines are circles around the rod axis in the electric transmitter, in the magnetic transmitter the *electric* force lines are circles around the normal of the frame antenna (at least for distances that are large compared to the frame). These statements are correct only for the vertical electric or magnetic dipole; for an oblique or horizontal position the circular symmetry is disturbed by the conductive ground. Generally speaking the data for the magnetic transmitter are deduced from those for the electric transmitter by replacing $\mathbf{E},\mathbf{H}$ by $\mathbf{H},-\mathbf{E}$, (for details see §35). Due to the boundary conditions for $\mathbf{E}$ (not for $\mathbf{H}$) in the case of an infinitely conductive ground, the signs in (16) and (17) are interchanged. Namely, for the magnetic Π_z (horizontal position of the plane of the frame), we have

$$\Pi_z = \frac{e^{ikR}}{R} - \frac{e^{ikR'}}{R'}\,, \qquad \Pi_z \equiv 0 \qquad \text{for } h \to 0 \tag{19}$$

and for the magnetic Π_x (vertical position of the plane of the frame) we have

$$\Pi_x = \frac{e^{ikR}}{R} + \frac{e^{ikR'}}{R'}, \qquad \Pi_x = 2\,\frac{e^{ikR}}{R} \qquad \text{for } h \to 0. \tag{20}$$

The proof will be given in exercise VI.1. The frame antenna of type (19) is of no practical importance, the antenna of type (20) will be treated in §35. As a transmitter this latter antenna shows a marked direction in the plane of the frame (e.g., for Π_x the y,z-plane) with the same characteristic as the electric rod antenna of (18). As a receiver it is arranged so that it can be rotated around the vertical line; if it is then oriented for maximal reception its plane points to the origin of the signal and it is therefore particularly suited for range finding (see §34).

§ 32. The Vertical Antenna Over an Arbitrary Earth

Let ε and σ be the electric constants of the ground. As regards its magnetic behavior we may assume $\mu = \mu_0$, which is sufficiently close to reality and simplifies the following calculations. We write

$$n^2 = \left(\varepsilon + i\,\frac{\sigma}{\omega}\right)\Big/\varepsilon_0 \tag{1}$$

and, as in optics, we call n the "complex refractive index." The wave number k of (31.6) will, in the following discussion, be called k_E in order to distinguish it from the wave number of air for which we keep the notation k. Then according to (31.6) and (31.2a) we have

$$k_E = nk. \tag{2}$$

We denote the altitude of the dipole antenna above the ground by h, as in (31.16).

We have to distinguish three regions:

I. Air $z > h$. In addition to the primary stimulation that becomes singular at the dipole $z = h$, $r = 0$, we have a secondary stimulation that is regular throughout due to currents induced in the ground. We write according to (31.14) and in analogy to (31.9)

$$\Pi_{\text{prim}} = \int_0^\infty I_0(\lambda r)\, e^{-\mu(z-h)}\,\frac{\lambda\, d\lambda}{\mu}, \qquad \Pi_{\text{sec}} = \int_0^\infty F(\lambda)\, I_0(\lambda r)\, e^{-\mu(z+h)} d\lambda. \tag{3}$$

where $F(\lambda)$ is, so to speak, the spectral distribution in the λ- continuum of the eigenfunctions, and is as yet undetermined. The factor $\exp(-\mu h)$ in the representation of Π_{sec} is convenient for what follows, and it is permissible since it is a pure function of λ and thus merely alters the meaning of $F(\lambda)$.

II. Air layer $h > z > 0$. Here, too, we have a primary and a secondary stimulation. Due to $z < h$ and according to the rule of signs of (31.14) we must write the former with a sign opposite to that in (3); the latter, being an analytic continuation, has the same form as in (3):

$$(4) \qquad \Pi_{\text{prim}} = \int_0^\infty I_0(\lambda r)\, e^{+\mu(z-h)} \frac{\lambda\, d\lambda}{\mu}, \; \Pi_{\text{sec}} = \int_0^\infty F(\lambda)\, I_0(\lambda r)\, e^{-\mu(z+h)}\, d\lambda .$$

Equations (3) and (4) insure the continuous behavior of the Π- field at the boundary between I and II for an arbitrary choice of $F(\lambda)$.

III. Earth $0 > z > -\infty$. Here there is no primary stimulation; the Π- field — denoted by Π_E — must be continuous throughout. In order to satisfy the differential equation for earth (31.3) with k_E^2 instead of k^2, we write:

$$(5) \qquad \Pi_E = \int_0^\infty F_E(\lambda)\, I_0(\lambda r)\, e^{+\mu_E z - \mu h}\, d\lambda , \qquad \mu_E^2 = \lambda^2 - k_E^2 .$$

According to our general rule we must choose the sign of $\mu_E z$ positive since $z < 0$. The factor $\exp(-\mu h)$ is adjoined for reasons of convenience; again, this merely influences the arbitrary function $F_E(\lambda)$. The functions $F_E(\lambda)$ and $F(\lambda)$ are determined from the boundary conditions on the surface of the earth.

According to Maxwell we must require the continuity of the tangential components of **E** and **H**. These are merely

$$\mathbf{E}_r \text{ and } \mathbf{H}_\varphi .$$

Indeed, the electric force lines are in the planes through the dipole axis, the magnetic force lines are circles around this axis, and hence $\mathbf{E}_\varphi$ and $\mathbf{H}_r$ vanish. (This follows from the fact that $\Pi = \Pi_z$ is a function of r and z alone.) Now according to (31.4) and (31.7) we have

$$(6) \qquad \begin{aligned} \mathbf{E}_r &= \frac{\partial}{\partial r}\frac{\partial \Pi}{\partial z}, \quad \mathbf{H}_\varphi = \frac{-k^2}{\mu_0 i\omega}\frac{\partial \Pi}{\partial r} \qquad \text{for } z > 0, \\ \mathbf{E}_r &= \frac{\partial}{\partial r}\frac{\partial \Pi_E}{\partial z}, \quad \mathbf{H}_\varphi = \frac{-k_E^2}{\mu_0 i\omega}\frac{\partial \Pi_E}{\partial r} \qquad \text{for } z < 0. \end{aligned}$$

Hence the continuity conditions for $z = 0$ are:

$$\frac{\partial}{\partial r}\frac{\partial \Pi}{\partial z} = \frac{\partial}{\partial r}\frac{\partial \Pi_E}{\partial z}, \qquad k^2 \frac{\partial \Pi}{\partial r} = k_E^2 \frac{\partial \Pi_E}{\partial r} .$$

We can integrate these conditions with respect to r and the constants of integration must be zero since all expressions vanish for $r \to \infty$. If in the second equation above we replace k_E^2 by $n^2 k^2$ according to (2) then we obtain:

$$\frac{\partial \Pi}{\partial z} = \frac{\partial \Pi_E}{\partial z}, \qquad \Pi = n^2 \Pi_E \qquad \text{for } z = 0. \tag{7}$$

On the right side of this equation we have to substitute the value of Π_E from (5) and on the left side we have to substitute the sum of Π_{prim} and Π_{sec} from (4). We thus obtain the conditions:

$$\int_0^\infty I_0(\lambda r)\, e^{-\mu h} (\lambda - \mu F - \mu_E F_E)\, d\lambda = 0, \tag{7 a}$$

$$\int_0^\infty I_0(\lambda r)\, e^{-\mu h} (\lambda + \mu F - n^2 \mu F_E)\, \frac{d\lambda}{\mu} = 0. \tag{7 b}$$

They are satisfied if we set

$$\mu F + \mu_E F_E = \lambda,$$
$$\mu F - n^2 \mu F_E = -\lambda.$$

Hence

$$F = \frac{\lambda}{\mu}\left(1 - \frac{2\mu_E}{n^2 \mu + \mu_E}\right), \qquad F_E = \frac{2\lambda}{n^2 \mu + \mu_E}. \tag{8}$$

Thus, we have demonstrated that equations (3), (4), (5) do indeed lead to a solution of our problem with its boundary conditions. The fact that there can be no other solution is deduced from the uniqueness axiom of physical boundary value problems, which always proves reliable. Due to the meaning of n, μ, μ_E equation (8) can be written in the more symmetric form:

$$\begin{aligned} F &= \frac{\lambda}{\sqrt{\lambda^2 - k^2}} \frac{k_E^2 \sqrt{\lambda^2 - k^2} - k^2 \sqrt{\lambda^2 - k_E^2}}{k_E^2 \sqrt{\lambda^2 - k^2} + k^2 \sqrt{\lambda^2 - k_E^2}}, \\ F_E &= \frac{2\lambda k^2}{k_E^2 \sqrt{\lambda^2 - k^2} + k^2 \sqrt{\lambda^2 - k_E^2}}. \end{aligned} \tag{8 a}$$

We again write the primary stimulation in its original form e^{ikR}/R with $R^2 = r^2 + (z - h)^2$ to show that the contribution to Π_{sec} that is due to the first term of F in (8) differs from Π_{prim} only by the fact that we have to replace $-h$ by $+h$, and hence R^2 by $R'^2 = r^2 + (z + h)^2$. Then representations (3) and (4) for regions I and II can be contracted and we obtain as *the general solution of our problem* for $z > 0$ and $z < 0$:

$$
(9)\qquad
\begin{aligned}
\Pi &= \frac{e^{ikR}}{R} + \frac{e^{ikR'}}{R'} - 2\int_0^\infty I_0(\lambda r)\, e^{-\mu(z+h)}\, \frac{\mu_E}{n^2\mu+\mu_E}\,\frac{\lambda\, d\lambda}{\mu},\\
\Pi_E &= 2\int_0^\infty I_0(\lambda r)\, e^{\mu_E z - \mu h}\, \frac{\lambda\, d\lambda}{n^2\mu+\mu_E}\,.
\end{aligned}
$$

If in particular we have $h = 0$ so that we can use equation (4) for the coinciding expressions e^{ikR}/R and $e^{ikR'}/R'$, then the first line of (9) can be rewritten in an elegant manner. According to previous work by the author we then have:

$$
(10)\qquad
\begin{aligned}
\Pi &= \int_0^\infty I_0(\lambda r)\, e^{-\mu z}\, \frac{2\,n^2\lambda\, d\lambda}{n^2\mu+\mu_E}\\
\Pi_E &= \int_0^\infty I_0(\lambda r)\, e^{\mu_E z}\, \frac{2\,\lambda\, d\lambda}{n^2\mu+\mu_E}\,.
\end{aligned}
$$

If, on the other hand, we consider the special case $|n| \to \infty$ of a completely conductive earth then μ_E can be neglected as compared to n^2 and the integrands in (9) will vanish. This confirms the result of the elementary reflection process in §31, equation (16):

$$
(10\,a)\qquad \Pi = \frac{e^{ikR}}{R} + \frac{e^{ikR'}}{R'}, \qquad \Pi_E = 0\,.
$$

It is profitable to consider this limit process with respect to n somewhat further. To this end we replace $n^2\mu + \mu_E$ by $n^2\mu$ in the denominator of the integrand in the first equation (9), and in the numerator we write, for all values of λ that are not too large,

$$
(10\,b)\qquad \mu_E = \sqrt{\lambda^2 - k_E^2} = k_E\sqrt{-1+\frac{\lambda^2}{k_E^2}} \sim -i\,k_E\,,
$$

and hence $\qquad \dfrac{\mu_E}{n} = -\,i\,k$

(concerning the sign of μ_E see the figure below). Then the first equation (9) becomes:

$$
(10\,c)\qquad \Pi = \frac{e^{ikR}}{R} + \frac{e^{ikR'}}{R'} + \frac{2ik}{n}\int_0^\infty I_0(\lambda r)\, e^{-\mu(z+h)}\,\frac{\lambda\, d\lambda}{\mu^2}\,.
$$

An intuitive interpretation of the latter integral[4] can be obtained as

[4] Since the denominator μ^2 vanishes at $\lambda = k$ the path of integration must be chosen in the complex λ-plane so as to avoid the point $\lambda = k$. This remark holds for the following λ-integrals, too. In the preceding integrals, starting with (31.14), we had the denominator μ, which did not destroy the convergence.

follows: corresponding to

$$R' = \sqrt{r^2 + (z+h)^2}, \quad \frac{e^{ikR'}}{R'} = \int_0^\infty I_0(\lambda r)\, e^{-\mu(z+h)} \frac{\lambda\, d\lambda}{\mu}$$

we write

$$R'' = \sqrt{r^2 + (z+h')^2}, \quad \frac{e^{ikR''}}{R''} = \int_0^\infty I_0(\lambda r)\, e^{-\mu(z+h')} \frac{\lambda\, d\lambda}{\mu}$$

and compute

$$\int_h^\infty \frac{e^{ikR''}}{R''}\, dh' = \int_h^\infty dh' \int_0^\infty \dots d\lambda = \int_0^\infty I_0(\lambda r)\, e^{-\mu z} \frac{\lambda\, d\lambda}{\mu} \int_h^\infty e^{-\mu h'}\, dh'$$
$$= \int_0^\infty I_0(\lambda r)\, e^{-\mu(z+h)} \frac{\lambda\, d\lambda}{\mu^2}.$$

hence the integral in (10c) stands for the action of an imaginary continuous covering of the ray $h < h' < \infty$ with dipoles that reach from the image point $z = -h$, $r = 0$ to $z = -\infty$, $r = 0$. Hence the approximating equation (10c) can also be written as:

(10d) $$\Pi = \frac{e^{ikR}}{R} + \frac{e^{ikR'}}{R'} + \frac{2ik}{n} \int_h^\infty \frac{e^{ikR''}}{R''}\, dh'.$$

In this connection we should remember a similar covering of a ray with imaginary source points that we used in a heat conduction problem in Fig. 15. While there we required exact satisfaction of the simple boundary condition $\partial u/\partial n + h\,u = 0$ (the h there, of course, had nothing to do with the h here), we now require approximate satisfaction of the complicated boundary conditions that arise from the juxtaposition of the air with the highly conductive earth.

From the above formulas we can deduce the field **E**, **H** by differentiation. However, we shall not write this somewhat cumbersome representation since we shall need it only in connection with the energy considerations of §36.

The integrals in (9) and (10) are not yet uniquely determined because of the square roots

(11) $$\mu = \sqrt{\lambda^2 - k^2}, \quad \mu_E = \sqrt{\lambda^2 - k_E^2}$$

that appear in them. Corresponding to the four combinations of signs of μ and μ_E, the integrand is four-valued, and its Riemann surface has four sheets. By our rule of signs in (31.14), which refers to the real part of and also applies to the real part of μ_E, one of the four sheets is singled out as a "permissible sheet." In order to insure the convergence of our integrals we demand that the path of integration at infinity shall be on the permissible sheet only. We achieve this by joining the "branch points"

$$\lambda = k \quad \text{and} \quad \lambda = k_E \tag{11a}$$

by two (essentially arbitrary) "branch cuts," which may not be intersected by the path of integration. Referring to Fig. 28 we therefore do not integrate along the real axis over the branch point $\lambda = k$, but avoid it by going into the negative imaginary half-plane and from there to infinity in, say, a direction parallel to the real λ-axis. Thus, along the path denoted by W_1 in Fig. 28 we integrate from $\lambda = 0$ to $\lambda = \infty$; this makes the meaning of the integrals in (9) and (10) precise.

But even then the representations (9) and (10) suffer from a mathematical inelegance: they are integrals with the fixed initial point $\lambda = 0$, not integrals along closed paths in the λ-plane,

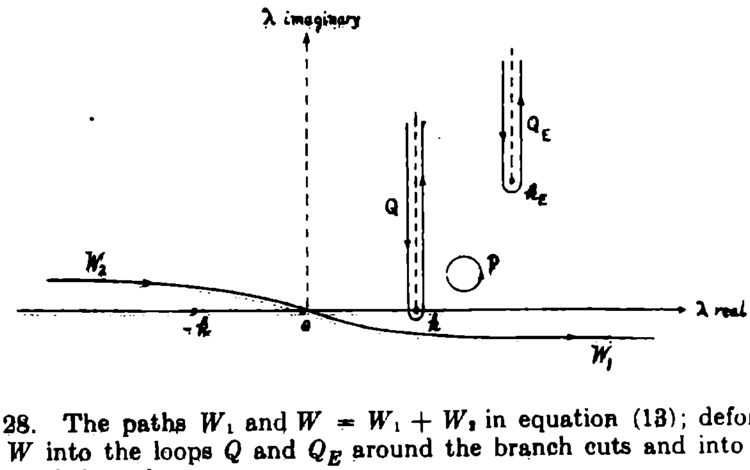


Fig. 28. The paths W_1 and $W = W_1 + W_2$ in equation (13); deformation of the path W into the loops Q and Q_E around the branch cuts and into the closed path P around the pole.

which, due to their deformability, would be much more useful. We remove this flaw by using the relation

$$I_0 = \frac{1}{2}\,(H_0^1 + H_0^2)$$

and the "semi-circuit relation" (10) of the introduction to exercise IV.2. If in the latter we set $\varrho = \lambda r$, then the preceding equation becomes

(12) $$I_0 = \frac{1}{2}\,[H_0^1(\lambda r) - H_0^1(\lambda e^{i\pi} r)].$$

We imagine (12) multiplied by an arbitrary function of λ^2 (indicated in the following by . . .) and by $\lambda\, d\lambda$ and integrated over W_1. Then, if we write $\lambda' = \lambda e^{i\pi}$, we obtain from the subtrahend on the right side of (12)

(12a) $$\int_{W'} H_0^1(\lambda' r) \ldots \lambda' d\lambda'.$$

Here W' is the path obtained from W_1 through reflection on the origin taken in the direction $\lambda' = 0 \rightarrow \lambda' = -\infty$, which, except for sign, is identical with the path W_2 in Fig. 28. Hence (12a) is the same as

(12b) $$\int_{W_2} H_0^1(\lambda' r) \ldots \lambda' d\lambda';$$

and from (12), if we denote the variable of integration throughout by λ and combine the paths W_1 and W_2 to $W = W_1 + W_2$, we obtain

(13) $$\int_{W_1} I_0(\lambda r) \ldots \lambda\, d\lambda = \frac{1}{2}\int_W H_0^1(\lambda r) \ldots \lambda\, d\lambda\,.$$

Thus we have achieved our purpose to replace the seemingly real integration that starts at $\lambda = 0$ in representations (9) and (10) by a complex integration over a path which closes at infinity. We consider this transformation (13) performed on all the integrals in (9) and (10). In particular we write, e.g., the primary stimulation of (31.14) and the first line of (10) in the new form

(14) $$\Pi_{\text{prim}} = \frac{1}{2}\int_W H_0^1(\lambda r)\, e^{-\mu|z|}\, \frac{\lambda\, d\lambda}{\mu},$$

(14a) $$\Pi \quad = \quad \int_W H_0^1(\lambda r)\, e^{-\mu z}\, \frac{n^2 \lambda\, d\lambda}{n^2 \mu + \mu_E}\,.$$

The attentive reader must have noticed long ago that Fig. 28 coincides with Fig. 26 (even with respect to the notation of the paths W, W_1 and the variable of integration λ), and that the present problem (determination of the function Π in space as subdivided by the surface of the earth for prescribed singularities at the dipole antenna) is summarized under the general *problem of Green's function.* Here we constructed the solution from eigenfunctions that satisfy the *radiation condition* at infinity. The fact that this condition is satisfied in the present case is made evident by the fact that in (14) and (14a) only the first Hankel function H^1 enters.[5]

We now consider the upper part of Fig. 28. Since we know that $H^1(\lambda r)$ vanishes in the infinite part of the positive imaginary half-plane, we can deform the path W into that half-plane. The path cannot be deformed across the branch cuts (11a), which it avoids by the loops Q and Q_E. However there is a further singularity of the integrand in (14) and in the analogous integrals, namely the point at which the denominator $n^2\mu + \mu_E$ vanishes. We denote it by

$$\lambda = p.$$

This corresponds to a *pole* of the integrand and must be avoided by the path of integration in a circuit P. We have not drawn the paths which join P to infinity since in the integration they cancel each other.

Of the three components Q, Q_E, P of the integral we can ignore the contribution of Q_E for large $|k_E|$, since $H^1(\lambda r)$ vanishes exponentially for great distances from the real axis. We first consider P separately, but we shall soon see that P and Q can hardly be separated.

From the defining relation for p

$$n^2\mu + \mu_E = 0 \tag{15}$$

we have

$$\sqrt{\frac{p^2 - k^2}{p^2 - k_E^2}} = -\frac{k^2}{k_E^2}, \qquad p^2 = \frac{k^2 k_E^2}{k^2 + k_E^2}, \tag{16}$$

which we can also write as

$$\frac{1}{p^2} = \frac{1}{k^2} + \frac{1}{k_E^2}. \tag{16a}$$

Due to $|k_E| \gg k$ we have approximately

$$p = k\left(1 - \frac{1}{2}\frac{k^2}{k_E^2}\right), \qquad k - p = \frac{k}{2}\frac{k^2}{k_E^2}. \tag{16b}$$

However, we wish to stress the fact that the precise value of p given by (16) or (16a) is symmetric in k and k_E.

[5] Here we have assumed a time dependence of the preferred form $\exp(-i\omega t)$. For a time dependence of the form $\exp(+i\omega t)$ we would have to make the transition from I to H^2 in (12) with the help of the semi-circuit relation (10a) in exercise (IV.2). Thus we would obtain a representation that, e.g., in (14a) is constructed from elements of the form

$$H_0^2(\lambda r)\, e^{-\mu z}\, e^{+i\omega t}$$

and hence also has the type of radiated waves.

We now compute the integral over P by applying the method of residues to (14a). Here we can let $\lambda = p$ in all the factors of the integrand of (14a), but we must replace the denominator which vanishes for $\lambda = p$ by

$$\frac{d}{d\lambda}(n^2\mu + \mu_E) = \lambda\left(\frac{n^2}{\sqrt{\lambda^2 - k^2}} + \frac{1}{\sqrt{\lambda^2 - k_E^2}}\right),$$

taken for $\lambda = p$. We thus obtain:

$$\frac{p}{k^2}K, \qquad K = \frac{k_E^2}{\sqrt{p^2 - k^2}} + \frac{k^2}{\sqrt{p^2 - k_E^2}}; \tag{17}$$

here the new quantity K is symmetric in k and k_E. Hence, as the contribution of P to (14a) we obtain:

$$\Pi = 2\pi i \frac{k_E^2}{K} H_0^1(pr)\, e^{-\sqrt{p^2 - k^2}\, z}. \tag{18}$$

In the same manner, for $z < 0$ (earth, interchange of k and k_E, and reversal of the sign of z) we obtain

$$\Pi_E = 2\pi i \frac{k^2}{K} H_0^1(pr)\, e^{+\sqrt{p^2 - k_E^2}\, z}. \tag{18a}$$

Except for the immediate neighborhood of the transmitter, namely, for all distances $|pr| \gg 1$ we can replace H by the asymptotic value (19.55). We then obtain

$$\Pi = 2\sqrt{\frac{2\pi i}{pr}}\, \frac{k_E^2}{K}\, e^{ipr - \sqrt{p^2 - k^2}\, z} \qquad z \geq 0, \tag{19}$$

$$\Pi_E = 2\sqrt{\frac{2\pi i}{pr}}\, \frac{k^2}{K}\, e^{ipr + \sqrt{p^2 - k_E^2}\, z} \qquad z \leq 0. \tag{19a}$$

These formulas bear all the marks of "*surface waves*," which are mentioned in v. II in connection with the water waves or the seismic Rayleigh waves, and which have the following properties:

1. They are tied to the surface $z = 0$ and decrease in both directions from that surface; in the direction of the earth they decrease rapidly due to the coefficient $(p^2 - k_E^2)^{\frac{1}{2}}$ of z; in the direction of the air the decrease is slow at first but exponential for large z.

2. The propagation along $z = 0$ is given by

$$\frac{dr}{dt} = \frac{\omega}{p},$$

and hence depends in a symmetric manner on the material constants air and earth, as must be the case for a surface wave.

3. If for the time being we neglect the absorption in the radial direction, then the amplitude of the expressions (19), (19a) decreases as $1/\sqrt{r}$, with increasing distance from the transmitter, whereas the intensity decreases as $1/r$. This too is a criterion for the essentially two-dimensional propagation of energy in the surface $z = 0$ (see p. 100).

4. For the sake of completeness we also mention the exponential absorption in the radial direction; it is given by the real part of ipr and according to (16b) and (1),(2) it is given by

$$-\frac{kr}{2}\,\mathrm{Re}\left(\frac{i}{n^2}\right) = -\frac{kr}{2}\,\frac{\varepsilon_0\sigma}{\omega}\Big/\left(\varepsilon^2+\frac{\sigma^2}{\omega^2}\right),$$

which is valid both for $z > 0$ and for $z < 0$.

For sufficiently large r, where the relative change of $r^{-\frac{1}{2}}$ is small, we can consider (19), (19a) as waves whose origin is at infinity, e.g., in the direction of the negative x-axis. These equations then become

(20) $$\Pi = A\,k_E^2\,e^{ipx-\sqrt{p^2-k^2}\,z},$$

(20a) $$\Pi_E = A\,k^2\,e^{ipx+\sqrt{p^2-k_E^2}\,z}$$

where A is a slowly varying amplitude factor, and hence represent the so-called "*Zenneck waves.*" As early as 1907 Zenneck,[6] in great graphical and numerical detail, investigated the fields **E**, **H** derived from (20), (20a), and discussed the material constants of the different types of soil (also fresh and salt water). It was the main point of the author's[7] work of 1909 to show that these fields are automatically contained in the wave complex, which, according to our theory, is radiated from a dipole antenna. This fact has, of course, not been changed. What has changed is the weight which we attached to it. At the time it seemed conceivable to explain the overcoming of the earth's curvature by radio signals with the help of the character of the *surface waves;* however, we know now that this is due to the ionosphere (see the introduction to this chapter). In any case the recurrent discussion in the literature on the "reality of the Zenneck waves" seems immaterial to us.

Epstein[8] has recently shown that the surface wave P taken by itself is a solution of our problem, and hence in principle does not have to be accompanied by the wave complex represented by Q. The latter, generally speaking, has the character of spatial waves and, in contrast to (20), is represented by the formal type

$$\Pi = B\,\frac{e^{ikr}}{r}.$$

[6] *Ann. Physik* 23, 846.
[7] *Ann. Physik* 28, 665.
[8] P. S. Epstein, *Proc. Natl. Acad. Sci. U. S.*, June 1947.

Under the actual circumstances of radio communication $P + Q$ is best represented by one contour integral which goes around the near points $\lambda = p$ and $\lambda = k$, and which must be discussed with the help of the saddle-point method. This has been carried out most completely by H. Ott.[9] However, we have to forego the presentation of his results in order not to get lost in the details of the problem.

We shall consider one more general aspect and one special formula which is convenient for numerical computations.

The general aspect concerns a kind of similarity relation of radio, the introduction of "numerical distance." Measured in terms of wavelengths the radial distance traversed by a spatial wave in the time t is kr (except for a factor 2π), the distance traversed by the surface wave in the same time is equal to the real part of pr. We form the difference of these distances and introduce the quantity

$$\varrho = i\,(k - p)\,r\,. \tag{21}$$

The absolute value of ϱ is called the *numerical distance.* The quantity ϱ is a pure number whose absolute value is small compared to kr. In fact, according to (16a) we have

$$|\varrho| \sim \frac{kr}{2}\frac{k^2}{|k_E|^2} = \frac{kr}{2|n|^2}\,. \tag{21a}$$

Hence, for small values of ϱ the spatial-wave type predominates in the expression of the reception intensity; in this case the ground peculiarities have no marked influence and we can make computations using an infinite ground conductivity without introducing great errors, as was done by Abraham (see §31). For larger ϱ the rivalry between spatial and surface waves becomes apparent, as the value of ϱ in (21) was defined in terms of the difference of the two propagations. In this case the material constants of the ground are important, and indeed not only σ, but also ε. Generally speaking, equal ϱ's imply equal wave types and equal reception strengths. Thus ϱ indicates a similarity relation. The fact that for sea water, due to its relatively high conductivity, ϱ is much smaller according to (21a) (for the same absolute distance r) than it is for fresh water or for an equally level dry soil, explains the good reception at sea (the difference in reception during the day and night is, of course, due to the ionosphere).

An expansion in ascending powers of ϱ led the author, in his first investigation (1909), to a convenient approximation formula, which since has been rededuced by different authors (B. Van der Pol, K. F.

[9] *Ann. Physik* **41**, 443 (1942).

Niessen, L. H. Thomas, F. H. Murray) partly in a simpler manner. In its final form the approximation formula reads:

$$\Pi = 2\,\frac{e^{ikr}}{r}\left(1 + i\sqrt{\pi\varrho}\,e^{-\varrho} - 2\sqrt{\varrho}\,e^{-\varrho}\int_0^{\sqrt{\varrho}} e^{\alpha^2}\,d\alpha\right), \tag{22}$$

It is valid for the air close to the earth.

In order to confirm the preceding remark we note: The first term, which is the dominating term for small ϱ, is of the spatial-wave type, and due to the factor 2 it corresponds to an infinitely conductive ground; the second term is of the surface-wave type and corresponds qualitatively, and for the purpose of our approximation even quantitatively, to the first equation (19); the third term represents the correction for larger ϱ. The generalization of (22) to the case of small finite distances z above the ground is

$$\Pi = 2\,\frac{e^{ikR}}{R}\left(1 + i\sqrt{\pi\varrho}\,e^{-\tau} - 2\sqrt{\varrho}\,e^{-\tau}\int_0^{\sqrt{\tau}} e^{\alpha^2}\,d\alpha\right); \tag{23}$$

where

$$\tau = i(k-p)\,r\left(1 + n\,\frac{z}{r}\right)^2;$$

for $z = 0$ we have $\tau = \varrho$ and (23) becomes the same as (22).

§ 33. The Horizontal Antenna Over an Arbitrary Earth

For a horizontal antenna lying in the x-direction it seems advisable to set the Hertz vector $\vec{\Pi}$ equal to Π_x. However, as we remarked at the end of §31 C, this is possible only for an infinitely conductive ground. We start by proving this fact.

For $\vec{\Pi} = \Pi_x$ we obtain from (31.4) and (31.7)

$$\begin{aligned} &\mathbf{E}_x = k^2\,\Pi_x + \frac{\partial^2\Pi_x}{\partial x^2}, \qquad \mathbf{E}_y = \frac{\partial^2\Pi_x}{\partial x\,\partial y} \qquad z \geq 0, \\ &\mathbf{E}_x = k_E^2\,\Pi_{xE} + \frac{\partial^2\Pi_{xE}}{\partial x^2}, \qquad \mathbf{E}_y = \frac{\partial^2\Pi_{xE}}{\partial x\,\partial y} \qquad z \leq 0. \end{aligned} \tag{1}$$

where $\mathbf{E}_x$ and $\mathbf{E}_y$ must be continuous at the boundary $z = 0$. From the above formulas for $\mathbf{E}_y$ we then deduce the continuity of Π_x, which implies the continuity of $\partial^2\Pi_x/\partial x^2$. But then the formulas for $\mathbf{E}_x$ imply the equality of k^2 and k_E^2, which is a contradiction.

We resolve this contradiction by writing the Hertz vector with *two components:*

(2) $$\vec{\Pi} = (\Pi_x,\ \Pi_z);$$

then instead of (1) we have

(3) $$\begin{aligned} \mathbf{E}_x &= k^2\, \Pi_x + \frac{\partial}{\partial x} \operatorname{div} \vec{\Pi}, & \mathbf{F}_y &= \frac{\partial}{\partial y} \operatorname{div} \vec{\Pi} & z \geq 0, \\ \mathbf{E}_x &= k_E^2\, \Pi_{xE} + \frac{\partial}{\partial x} \operatorname{div} \vec{\Pi}_E, & \mathbf{E}_y &= \frac{\partial}{\partial y} \operatorname{div} \vec{\Pi}_E & z \leq 0. \end{aligned}$$

Hence for $z = 0$:

(4) $$\operatorname{div} \vec{\Pi} = \operatorname{div} \vec{\Pi}_E$$

and

(5) $$k^2\, \Pi_x = k_E^2\, \Pi_{xE}\,.$$

For the magnetic components, according to (31.4) and (31:7), we have

$$\begin{aligned} \mathbf{H}_x &= \frac{k^2}{i\mu_0\omega} \frac{\partial \Pi_z}{\partial y}, & \mathbf{H}_y &= \frac{k^2}{i\mu_0\omega}\left(\frac{\partial \Pi_x}{\partial z} - \frac{\partial \Pi_z}{\partial x}\right) & z \geq 0, \\ \mathbf{H}_x &= \frac{k_E^2}{i\mu_0\omega} \frac{\partial \Pi_{zE}}{\partial y}, & \mathbf{H}_y &= \frac{k_E^2}{i\mu_0\omega}\left(\frac{\partial \Pi_{xE}}{\partial z} - \frac{\partial \Pi_{zE}}{\partial x}\right) & z \leq 0. \end{aligned}$$

From the continuity of $\mathbf{H}_x$ it follows that

(6) $$k^2\, \Pi_z = k_E^2\, \Pi_{zE}$$

and from the continuity of $\mathbf{H}_y$ it follows that

(7) $$k^2 \frac{\partial \Pi_x}{\partial z} = k_E^2 \frac{\partial \Pi_{xE}}{\partial z}\,.$$

Hence we have two conditions (5) and (7) for Π_x which we can write in the form

(8) $$\Pi_x = n^2\, \Pi_{xE}, \qquad \frac{\partial \Pi_x}{\partial z} = n^2 \frac{\partial \Pi_{xE}}{\partial z},$$

After we have determined Π_x we obtain the two conditions (6) and (4) for Π_z:

(9) $$\Pi_z = n^2\, \Pi_{zE}, \qquad \frac{\partial \Pi_z}{\partial z} - \frac{\partial \Pi_{zE}}{\partial z} = \frac{\partial \Pi_{xE}}{\partial x} - \frac{\partial \Pi_x}{\partial x}.$$

The computation of Π_x is carried out by the methods of §32. We again distinguish the three regions:

I. $\infty > z > h,$ II. $h > z > 0,$ III. $0 > z > -\infty,$

In I and II the function Π_z is composed of a primary and a secondary stimulation, which can be expressed exactly as in (32.3); in III we have only the secondary stimulation of the form (32.5). The conditions (8) then yield, in analogy to (32.7a,b),

$$\int_0^\infty I_0(\lambda r)\, e^{-\mu h}(\lambda - \mu F - n^2 \mu_E F_E)\, d\lambda = 0, \tag{10a}$$

$$\int_0^\infty I_0(\lambda r)\, e^{-\mu h}(\lambda + \mu F - n^2 \mu F_E)\, \frac{d\lambda}{\mu} = 0 \tag{10b}$$

and by setting the parentheses equal to zero we obtain:

$$F = \frac{\lambda}{\mu}\left(-1 + \frac{2\mu}{\mu + \mu_E}\right), \quad F_E = \frac{1}{n^2}\, \frac{2\lambda}{\mu + \mu_E}\,. \tag{11}$$

This expression for F is written so that the second term vanishes for $n \to \infty$, which is the same as $\mu_E \to \infty$, so that in the limit only the first term $F = -\lambda/\mu$ remains. If we substitute (11) in the equations (32.3,4,5), then in analogy to (32.9) we obtain the representation of the Π_z- field:

$$\begin{aligned} \Pi_z &= \frac{e^{ikR}}{R} - \frac{e^{ikR'}}{R'} + 2\int_0^\infty I_0(\lambda r)\, e^{-\mu(z+h)}\, \frac{\lambda\, d\lambda}{\mu + \mu_E}\,, \\ \Pi_{zE} &= \frac{2}{n^2}\int_0^\infty I_0(\lambda r)\, e^{+\mu_E z - \mu h}\, \frac{\lambda\, d\lambda}{\mu + \mu_E}\,. \end{aligned} \tag{12}$$

where $R^2 = r^2 + (z-h)^2$, $R'^2 = r^2 + (z+h)^2$.

If in particular $h = 0$ then we have $R' = R$ and (12) simplifies to:

$$\begin{aligned} \Pi_z &= 2\int_0^\infty I_0(\lambda r)\, e^{-\mu z}\, \frac{\lambda\, d\lambda}{\mu + \mu_E}\,, \\ \Pi_{zE} &= \frac{2}{n^2}\int_0^\infty I_0(\lambda r)\, e^{+\mu_E z}\, \frac{\lambda\, d\lambda}{\mu + \mu_E}\,. \end{aligned} \tag{12a}$$

If on the other hand we consider the special case $n \to \infty$, then we also have $|\mu_E| \to \infty$ and the integrals in (12) vanish, so that (12) reduces to

$$\Pi_z = \frac{e^{ikR}}{R} - \frac{e^{ikR'}}{R'}\,, \qquad \Pi_{zE} = 0 \tag{12b}$$

in agreement with (31.17).

The integration in equations (12) and (12a) is to be taken over the path W_1 of Fig. 28. Here again we can profitably replace this path by the closed path $W = W_1 + W_2$. If at the same time we replace I_0 by $\frac{1}{2} H_0^1$ then for vanishing h but finite k_E we obtain, in analogy to (32.14a),

$$(12\text{c}) \quad \Pi_x = \int\limits_W H_0^1(\lambda r)\, e^{-\mu z} \frac{\lambda\, d\lambda}{\mu + \mu_E}, \quad \Pi_{xE} = \frac{1}{n^2} \int\limits_W H_0^1(\lambda r)\, e^{+\mu_E z} \frac{\lambda\, d\lambda}{\mu + \mu_E}.$$

We now turn to the determination of Π_z and first consider the second condition (9). Since Π_x and Π_{xE} do not depend on x and y separately but only on $r = \sqrt{x^2 + y^2}$, we have

$$\frac{\partial \Pi_x}{\partial x} = \frac{\partial \Pi_x}{\partial r} \frac{\partial r}{\partial x} = \cos\varphi \frac{\partial \Pi_x}{\partial r}, \qquad \varphi = \sphericalangle (x, \vec{r}),$$

and a corresponding relation for $\partial \Pi_{xE}/\partial x$. From the second equation (9) it follows that Π_z must also contain the factor $\cos\varphi$. Hence we deduce that Π_z can no longer be constructed from the eigenfunctions $I_0(\lambda r)\, e^{\mp \mu z}$; it is necessary to use Bessel functions with the next higher index 1

$$I_1(\lambda r) \cos\varphi\, e^{\mp \mu z}$$

Considering the fact that Π_z should contain no primary stimulation we write:

$$(13) \quad \begin{aligned} \Pi_z &= \cos\varphi \int I_1(\lambda r)\, e^{-\mu(z+h)}\, \Phi(\lambda)\, d\lambda, \\ \Pi_{zE} &= \cos\varphi \int I_1(\lambda r)\, e^{+\mu_E z - \mu h}\, \Phi_E(\lambda)\, d\lambda. \end{aligned}$$

where Φ and Φ_E are still to be determined. The first condition (9) then yields

$$(13\text{a}) \qquad \Phi = n^2 \Phi_E.$$

The second condition (9) yields

$$(13\text{b}) \quad \begin{aligned} &-\cos\varphi \int I_1(\lambda r)\, e^{-\mu h} (\mu \Phi + \mu_E \Phi_E)\, d\lambda \\ &= \cos\varphi \left(\frac{1}{n^2} - 1\right) \int I_0'(\lambda r)\, e^{-\mu h} \frac{2\lambda^2\, d\lambda}{\mu + \mu_E}. \end{aligned}$$

In fact, in the representation (12) the terms not under the integral signs vanish for $z = 0$. If we multiply the numerator and denominator of the integrand on the right side by $\mu - \mu_E$ and consider the fact that

$$\mu^2 - \mu_E^2 = k_E^2 - k^2 = k_E^2 \left(1 - \frac{1}{n^2}\right)$$

and that according to (19.52b) we have $I_0'(\varrho) = -I_1(\varrho)$, then we can contract (13b) to

$$(13\text{c}) \quad -\cos\varphi \int I_1(\lambda r)\, e^{-\mu h}\left(\mu\Phi + \mu_E\Phi_E + \frac{2}{k_E^2}(\mu - \mu_E)\lambda^2\right) d\lambda = 0 .$$

From this we deduce a further relation between Φ and Φ_E :

$$(13\text{d}) \qquad \mu\Phi + \mu_E\Phi_E = -\frac{2}{k_E^2}(\mu - \mu_E)\lambda^2 .$$

This, together with (13a) and (32.2), yields

$$(14) \qquad \Phi = -\frac{2\lambda^2}{k^2}\frac{\mu - \mu_E}{n^2\mu + \mu_E}, \qquad \Phi_E = \frac{\Phi}{n^2} .$$

According to (13) the final representation of Π_z is then

$$(15) \qquad \begin{aligned} \Pi_z &= -\frac{2}{k^2}\cos\varphi \int I_1(\lambda r)\, e^{-\mu(z+h)}\frac{\mu - \mu_E}{n^2\mu + \mu_E}\lambda^2\, d\lambda , \\ \Pi_{zE} &= -\frac{2}{k_E^2}\cos\varphi \int I_1(\lambda r)\, e^{+\mu_E z - \mu h}\frac{\mu - \mu_E}{n^2\mu + \mu_E}\lambda^2\, d\lambda . \end{aligned}$$

The path of integration is W_1 of Fig. 28, or, if we replace I_1 by $\frac{1}{2}H_1^1$, the path $W = W_1 + W_2$. Since the denominator in (15) coincides with that of Π_z in §32, "surface waves" also exist for the Π_z which is induced by a horizontal antenna. These surface waves correspond to the pole P in Fig. 28 and they are superimposed on the "spatial waves" or merge with them. Under the assumption $k_E \to \infty$, which implies $n \to \infty, \mu_E \to \infty$, the component Π_z vanishes. Hence the induced vertical component is strongly dependent on the nature of the ground and thus does not appear in the previous elementary treatment of §31.

A principal distinction of the horizontal antenna as compared to the vertical antenna is its *directed radiation*, which is implied by the factor $\cos\varphi$ in (15). The same factor is contained in the electric and magnetic field components which determine the radiation and it is a quadratic factor of the radiated energy. Later on we shall see that the component Π_x , which is free of $\cos\varphi$, in general gives no essential contribution to distant transmissions, and hence it can be neglected in the following discussion.

The solid curve in Fig. 29 represents the "direction characteristic" of the horizontal antenna. In order to obtain this curve we plot the radiated energy $\sqrt{E} = M\cos\varphi$ in a polar diagram, where M is the maximum of $\sqrt{E}$ radiated in the direction $\varphi = 0$. This curve is symmetric with respect to the direction $\varphi = \pm \pi/2$ in which there is no radiation; the radiation in the forward direction $\varphi = 0$ and in the

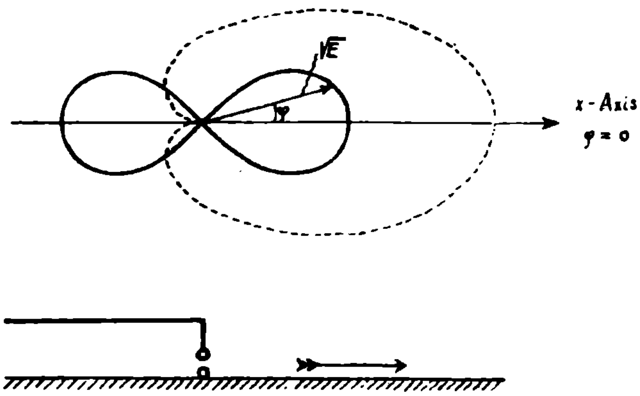


Fig. 29. Upper half: solid curve = direction characteristic of the horizontal antenna; broken curve = direction characteristic of the Marconi antenna.
Lower half: diagram of the Marconi antenna.

backward direction $\varphi = \pi$ is the same. If we combine the horizontal antenna *coherently* with a vertical antenna so that the vertical antenna alone would give the same radiation M as would be given by the horizontal antenna in the direction $\varphi = 0$ (the polar diagram would be a circle of radius M), then, we obtain as the total characteristic the curve

$$\sqrt{E} = M(1 + \cos\varphi) = \begin{cases} 2M & \text{for } \varphi = 0 \\ M & \text{for } \varphi = \pi/2 \\ 0 & \text{for } \varphi = \pi. \end{cases}$$

This characteristic is represented by the broken curve and shows a stronger directedness than the solid horizontal antenna curve.

In the lower half of Fig. 29 we sketched an arrangement by which such a combination of horizontal and vertical antennas was realized on a large scale by Marconi (about 1906) for transatlantic communication (station Clifden in Ireland). The preferred radiation in the direction of the arrow in Fig. 29 aroused general amazement and raised the problem studied by H. von Hörschelmann,[10] in which the above theory was developed (Marconi worked only with the instinct of the ingenious experimenter). However the Clifden arrangement was somewhat cumbersome, and it has since been replaced by a more convenient combination of two or more vertical antennas (see Fig. 30).

In Fig. 30 we have drawn a horizontal antenna of the effective

[10] Dissertation, Munich 1911, *Jahresber. f. drahtl. Tel.* 5, 14, 158 (1912).

length l, together with the current which flows through and the influx and outflux through the earth. The last two are equivalent to two

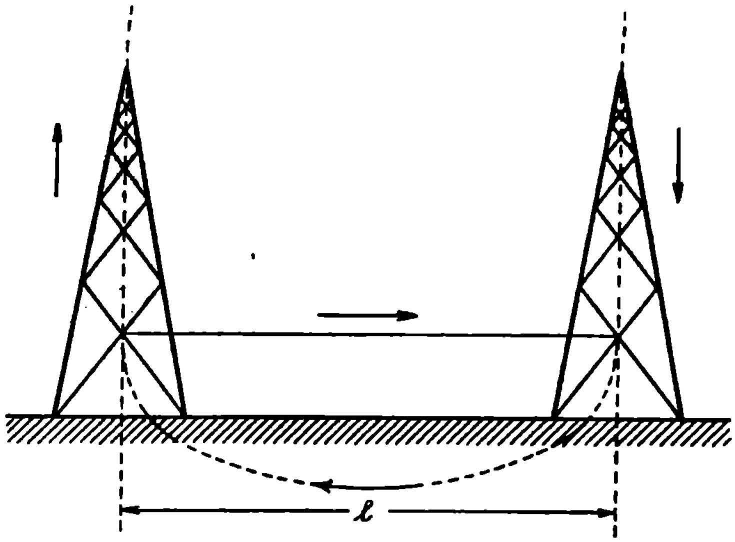


Fig. 30. Horizontal antenna with the accompanying earth currents; at a distance the effect of the two vertical antennas is the same as the effect of the horizontal antenna.

coherent vertical antennas of opposite phase, which we have indicated by towers. Their action at a distance is represented by a formula of the type

$$(16) \qquad l \cos \varphi \frac{\partial \Pi_1}{\partial r}, \qquad \Pi_1 = \frac{e^{ikR}}{R},$$

where Π_1 is the Hertz vector of the individual tower. We want to show that our theory in a rough approximation really leads to a formula of this type.

Since we are interested only in action at a distance we set $h = 0$ in (15) and in analogy to (32.10b) write

$$\mu - \mu_E \sim -\mu_E = i\,k\,n, \qquad n^2 \mu + \mu_E \sim n^2 \mu .$$

In addition we have the relation

$$\frac{\partial}{\partial r} I_0(\lambda r) = -\lambda I_1(\lambda r).$$

The first equation (15) thus becomes

$$(16\text{a}) \qquad \Pi_z = \frac{2i}{kn} \cos \varphi \frac{\partial}{\partial r} \int_0^\infty I_0(\lambda r)\, e^{-\mu z} \frac{\lambda\, d\lambda}{\mu},$$

so that we now have the primary stimulation e^{ikR}/R under the integral sign. Hence, we have actually obtained the form of equation (16); for the length of the antenna l we obtain

$$|l| = \frac{2}{k\,|n|} = \frac{2\sqrt{\varepsilon_0}}{k\,|\sqrt{\varepsilon + i\,\sigma/\omega}|}\,. \tag{16b}$$

Due to the meaning of k this length $|\,l\,|$ is of the order of magnitude of the wave length λ, but it also depends strongly on the nature of the ground; in the limit $\sigma \to \infty$ we have $l = 0$ as has been stressed before.

The same approximation method leads to an estimate of the order of magnitude of Π_x. We start from the first equation (12) and set $h = 0$ as well as $\mu + \mu_E \sim \mu_E \sim -i\,k\,n$. We then obtain

$$\begin{aligned}\Pi_x = \frac{2\,i}{k\,n}\int\limits_{W_1} I_0(\lambda\,r)\,e^{-\mu z}\,\lambda\,d\lambda &= -\frac{2\,i}{k\,n}\frac{\partial}{\partial z}\int\limits_{W_1} I_0(\lambda\,r)\,e^{-\mu z}\,\frac{\lambda\,d\lambda}{\mu} \\ &= -\frac{2\,i}{k\,n}\frac{\partial}{\partial z}\frac{e^{ikR}}{R}.\end{aligned} \tag{16c}$$

Now we have

$$\frac{\partial}{\partial z}\frac{e^{ikR}}{R} = \frac{z}{R}\frac{d}{dR}\frac{e^{ikR}}{R}, \quad \frac{\partial}{\partial r}\frac{e^{ikR}}{R} = \frac{r}{R}\frac{d}{dR}\frac{e^{ikR}}{R}.$$

The ratio of these latter quantities is z/r, and hence is very small in the neighborhood of the surface of the earth at a great distance from the transmitter. According to (16c) and (16a) $-\Pi_x$ and Π_z have the same ratio. Hence we have

$$|\Pi_x| \ll |\Pi_z|\,. \tag{17}$$

This fact has been mentioned before but is proved here for the first time.

The result is very remarkable: *The primary stimulation* Π_x *serves only to give rise to the secondary stimulation* Π_z. *The transmission at a distance is caused by* Π_z *alone.* Only in the immediate neighborhood of the transmitter, due to the prescribed pole of Π_x, does Π_z have an effect which outweighs that of Π_x. *At a great distance the field of transmission of a horizontal antenna has the same character as the field of transmission of a vertical antenna,* except for the φ-dependence which indicates the primary origin from a horizontal antenna. *In both cases the signals for large distances are best received with a vertical antenna;* a horizontal antenna would be unsuited as a receiver, since the horizontal component of the induced field is always small compared to the vertical component, even for a moderately conductive ground.

These results are generally known in practice, but they can hardly be understood without our theory which takes the nature of the soil into account.

We note that approximations (16a) and (16c) can be considered as the first terms of an expansion in ascending powers of the numerical distance ϱ. Just as we had to complement the term e^{ikR}/R by terms dependent on ϱ for the vertical antenna, so now we must correct (16a) and (16c) by terms dependent on ϱ.

§ 34. Errors in Range Finding for an Electric Horizontal Antenna

In navigation, range finding means the location of that direction from which a signal reaches the receiver. As an ideal *receiver* for radio signals we have the frame antenna, which was described at the end of §31, and which will be investigated in greater detail in §35. We consider the receiving antenna rotatable around a vertical axis. As for navigation, we assume the receiver to be at sea, near the surface of the earth. We assume the *transmitter* to be a horizontal antenna. Then, corresponding to the directional characteristic of Fig. 29, we not only expect maximal reception on all points of the x-axis for an x-directed transmitter, but at every point (x,y) on the earth we expect a maximal reception in the r-direction from which the signal comes, and no reception in the φ-direction. In reality things are not that simple because, in addition to the principal radiation of the order $1/r$, the horizontal antenna also emits radiation of the order $1/r^2$.

In order to prove this last fact we have to carry the approximation of the field one step further than we did in the equations of the preceding section. Namely, equations (33.16a) and (33.16c) yield $\operatorname{div} \vec{\Pi} = 0$ and hence, since $\Pi_x = 0$, for $h = 0$ and $z = 0$ they yield a field $\vec{\mathbf{E}}_z$ perpendicular to the surface of the earth. We now compute $\operatorname{div} \vec{\Pi}$ with greater precision. We obtain Π_x from (33.12c) with $h = 0$, and Π_z from (33.15) by setting $h = 0$ and replacing I_1 by $\frac{1}{2} H_1^1$. Then we obtain

$$\frac{\partial \Pi_x}{\partial x} = -\cos\varphi \int_W H_1^1(\lambda r)\, e^{-\mu z} \frac{\lambda^2 d\lambda}{\mu + \mu_E},$$

$$\frac{\partial \Pi_z}{\partial z} = \cos\varphi \int_W H_1^1(\lambda r)\, e^{-\mu z} \frac{\mu}{k^2} \frac{\mu - \mu_E}{n^2\mu + \mu_E} \lambda^2 d\lambda,$$

$$\operatorname{div} \vec{\Pi} = -\cos\varphi \int_W H_1^1(\lambda r)\, e^{-\mu z} \left(\frac{1}{\mu + \mu_E} - \frac{\mu}{k^2} \frac{\mu - \mu_E}{n^2\mu + \mu_E}\right) \lambda^2 d\lambda;$$

and by a simple contraction:

$$(1)\qquad \begin{aligned} \operatorname{div} \vec{\Pi} &= -\cos\varphi \int_W H_1^1(\lambda r)\, e^{-\mu z} \frac{\lambda^2 d\lambda}{n^2\mu + \mu_E} \\ &= \cos\varphi \frac{\partial}{\partial r} \int_W H_0^1(\lambda r)\, e^{-\mu z} \frac{\lambda\, d\lambda}{n^2\mu + \mu_E}\,. \end{aligned}$$

According to (32.14a) the last integral is nothing else than the Π-field of a *vertical* antenna divided by n^2. Since we assume $z = 0$ we may represent this field by (32.22). It even suffices to use the first term of (32.22), which we can write as

$$(2)\qquad \operatorname{div} \vec{\Pi} = \frac{2}{n^2} \cos\varphi \frac{\partial}{\partial r} \frac{e^{ikr}}{r}\,.$$

It is now profitable to use polar coordinates. Then we obtain from (2), if we neglect the terms with $(kr)^{-3}$,

$$(3)\qquad \begin{aligned} \operatorname{grad}_r \operatorname{div} \vec{\Pi} &= \frac{\partial}{\partial r} \operatorname{div} \vec{\Pi} = \frac{2}{n^2} \cos\varphi \frac{\partial^2}{\partial r^2} \frac{e^{ikr}}{r} \\ &= -\frac{2k^2}{n^2} \cos\varphi \left(1 - \frac{2}{ikr}\right) \frac{e^{ikr}}{r}\,, \\ \operatorname{grad}_\varphi \operatorname{div} \vec{\Pi} &= \frac{1}{r} \frac{\partial}{\partial\varphi} \operatorname{div} \vec{\Pi} = -\frac{2}{n^2} \sin\varphi \frac{1}{r} \frac{\partial}{\partial r} \frac{e^{ikr}}{r} \\ &= +\frac{2k^2}{n^2} \frac{\sin\varphi}{ikr} \frac{e^{ikr}}{r}\,; \end{aligned}$$

and from (31.4) we obtain:

$$(4)\qquad \begin{aligned} \mathbf{E}_r &= \cos\varphi\, \mathbf{E}_x + \sin\varphi\, \mathbf{E}_y = k^2 \cos\varphi\, \Pi_x + \operatorname{grad}_r \operatorname{div} \vec{\Pi}\,, \\ \mathbf{E}_\varphi &= -\sin\varphi\, \mathbf{E}_x + \cos\varphi\, \mathbf{E}_y = -k^2 \sin\varphi\, \Pi_x + \operatorname{grad}_\varphi \operatorname{div} \vec{\Pi}\,. \end{aligned}$$

We still have to estimate Π_x. With the approximation in (33.16c) we would obtain $\Pi_x = 0$ for $z = 0$; a more exact computation yields, if we again ignore the terms with $(kr)^{-3}$,

$$(5)\qquad k^2 \Pi_x = \frac{2}{n^2} \frac{1}{r} \frac{\partial}{\partial r} \frac{e^{ikr}}{r} = -\frac{2k^2}{n^2} \frac{1}{ikr} \frac{e^{ikr}}{r}\,.$$

Hence we obtain from (3),(4),(5)

$$\begin{aligned} \mathbf{E}_r &= -\frac{2k^2}{n^2} \cos\varphi \left(1 - \frac{1}{ikr}\right) \frac{e^{ikr}}{r}\,, \\ \mathbf{E}_\varphi &= +\frac{4k^2}{n^2} \frac{\sin\varphi}{ikr} \frac{e^{ikr}}{r}\,; \end{aligned}$$

and thus, due to $kr \gg 1$,

$$\text{(6)} \qquad \begin{aligned} \mathbf{E}_r &= -\frac{2k^2}{n^2}\cos\varphi\,\frac{e^{ikr}}{r}, \\ \mathbf{E}_\varphi &= +\frac{4k^2}{n^2}\,\frac{\sin\varphi}{ikr}\,\frac{e^{ikr}}{r}. \end{aligned}$$

From this we conclude that for $\varphi = 0$ we have $\mathbf{E}_\varphi = 0$ and that the horizontal antenna field is in the r-direction. Therefore —

A rotatable frame antenna situated *on the extension of the transmitting antenna* shows the strongest reception *in the direction of the transmitting antenna,* as we had expected from the start.

On the other hand for $\varphi = \pm\pi/2$ we have

$$\text{(7)} \qquad \mathbf{E}_r = 0, \qquad \mathbf{E}_\varphi = \frac{4k^2}{n^2}\,\frac{e^{ikr}}{ikr^2}.$$

A rotatable receiving antenna situated *on the perpendicular to the transmitting antenna* shows a misdirection. In the position of maximal reception the receiving antenna does not point in the direction of the transmitting antenna, but in a direction perpendicular to it, which is *parallel to the transmitting antenna.* However, the reception is very weak, being of the order $1/r^2$; this explains the fact that in Fig. 29, where we considered only terms of the order $1/r$, this reception was *zero.*

Generally we may denote $\mathbf{E}_r$ as "correct direction" and $\mathbf{E}_\varphi$ as "misdirection." The latter, as in (7), is entirely due to terms of the order $1/r^2$.

For an arbitrary φ the "relative misdirection" in our approximation is, according to (6),

$$\left|\frac{\mathbf{E}_\varphi}{\mathbf{E}_r}\right| = \frac{2}{kr}\tan\varphi.$$

It increases to infinity as φ approaches $\pi/2$, which means that for that value the correct direction vanishes, corresponding to $\mathbf{E}_r = 0$ in (7).

The practical engineer is in error if he considers such misdirections the result of mistakes in the construction of the transmitting or the receiving antenna. As we have seen these misdirections are in the nature of things. Certain other misdirections called "after effects," which are due to reflections on the ionosphere, will not be discussed here.

§ 35. The Magnetic or Frame Antenna

The *frame antenna* can be used not only for range finding but also for directed transmission. In both cases the plane of the loop is taken perpendicular to the surface of the earth and the normal to this plane will be taken as the x-axis. For rectangular forms the loop consists of two pairs of coherent vertical and horizontal antennas of opposite phase, similar to the scheme in Fig. 30.

In §31 D we called such an antenna *magnetic*, no matter what the shape of the loop. Our frame antenna, which is situated in the y,z-plane, is equivalent to a *magnetic dipole* in the $\vec{x}$-direction; its primary action can be represented by a Hertz vector $\vec{\Pi}_{\text{prim}} = \Pi_x$. Due to the presence of the earth this Hertz vector becomes a general vector $\vec{\Pi}$.

The relation between $\vec{\Pi}$ and the electromagnetic field in a vacuum is the same as in (31.4), *but we must replace* **E**, **H**, ε_0, μ_0 by **H**, $-$**E**, μ_0, ε_0 In fact this interchange transforms the Maxwell equations (31.5) into themselves. Thus, in a vacuum, as counterpart to (31.4) we have:

$$\text{(1)} \qquad \mathbf{H} = k^2 \vec{\Pi} + \text{grad div}\, \vec{\Pi}, \quad -\mathbf{E} = \frac{k^2}{\varepsilon_0 i \omega} \text{curl}\, \vec{\Pi} = -\mu_0 i \omega \,\text{curl}\, \vec{\Pi}$$

and in the earth, as counterpart to (31.7) we have:

$$\text{(2)} \qquad \mathbf{H} = k_E^2 \vec{\Pi} + \text{grad div}\, \vec{\Pi}; \quad -\mathbf{E} = \frac{k_E^2}{\varepsilon i \omega} \text{curl}\, \vec{\Pi},$$

where we have as before:

$$\text{(2a)} \qquad k_E^2 = \varepsilon \mu_0 \omega^2 + i \sigma \mu_0 \omega, \quad k^2 = \varepsilon_0 \mu_0 \omega^2 = \omega^2/c^2 .$$

The vector Π again satisfies the differential equation (31.3).

The boundary conditions for $z = 0$ force us to consider $\vec{\Pi}$ as a vector with two components

$$\vec{\Pi} = (\Pi_x, \Pi_z).$$

just as in the case of the electric horizontal antenna. Indeed, we have

$$\text{(3)} \qquad \Pi_z = \Pi_{zE}, \qquad \qquad \text{(4)} \qquad \frac{\partial \Pi_x}{\partial z} = \frac{\partial \Pi_{xE}}{\partial z}$$

due to the continuity of $\mathbf{E}_{\text{tang}}$,

$$\text{(5)} \quad \text{div}\, \vec{\Pi} = \text{div}\, \vec{\Pi}_E, \qquad \qquad \text{(6)} \quad k^2 \Pi_x = k_E^2 \Pi_{xE}$$

due to the continuity of $\mathbf{H}_{\text{tang}}$.

Hence, we have two conditions (4) and (6) for Π_x, and two further conditions (3) and (5) that determine Π_z from the known Π_x. Conditions (4) and (6) are exactly the same as the conditions (32.7) for the vertical antenna. Hence, we can apply the previous representations (32.9) *et seq.* directly to our Π_x. Written in the form (32.14a) as specialized for $h = 0$, these representations read:

$$(7) \qquad \begin{aligned} \Pi_x &= \int_W H_0^1(\lambda r)\, e^{-\mu z} \frac{n^2 \lambda\, d\lambda}{n^2 \mu + \mu_E}, \\ \Pi_{xE} &= \int_W H_0^1(\lambda r)\, e^{+\mu_E z} \frac{\lambda\, d\lambda}{n^2 \mu + \mu_E}. \end{aligned}$$

For the same reasons as in the case of the electric horizontal antenna, we write Π_z in the form (33.13) that contains $\cos\varphi$. However, due to condition (3), the functions Φ and Φ_E will now be equal; their common value is determined from (5):

$$\Phi = \frac{\mu - \mu_E}{n^2 \mu + \mu_E} \frac{\lambda^2}{k^2}.$$

Hence by setting $h = 0$ and replacing I by H we obtain from (33.15)

$$(8) \qquad \begin{aligned} \Pi_z &= -\frac{\cos\varphi}{k^2} \int_W H_1^1(\lambda r)\, e^{-\mu z} \frac{\mu - \mu_E}{n^2 \mu + \mu_E} \lambda^2\, d\lambda, \\ \Pi_{zE} &= -\frac{\cos\varphi}{k^2} \int_W H_1^1(\lambda r)\, e^{+\mu_E z} \frac{\mu - \mu_E}{n^2 \mu + \mu_E} \lambda^2\, d\lambda. \end{aligned}$$

However, in contrast to the electric horizontal antenna, we may now neglect Π_z as compared to Π_x, so that in the discussion of the field and of its directional characteristic we shall consider the component Π_x alone.

According to (1) we then have

$$(9) \qquad \mathbf{E}_x = 0, \quad \mathbf{E}_y = \mu_0 i \omega \frac{\partial \Pi_x}{\partial z}, \quad \mathbf{E}_z = -\mu_0 i \omega \frac{\partial \Pi_x}{\partial y}.$$

Now we obtained the first line of (7) from the representation (32.14a), which for small numerical distances was approximated by (32.23). Applying the latter to (7) we obtain

$$(10) \qquad \Pi_x = 2 \frac{e^{ikR}}{R} (1 + \cdots), \qquad R = \sqrt{r^2 + z^2}.$$

This agrees with the representation (31.20) for an infinitely conductive ground. From (9) and (10) for $z = 0$, we now obtain

$$\mathbf{E}_x = \mathbf{E}_y = 0, \qquad \mathbf{E}_z = -2 \mu_0 i \omega \frac{y}{r} \frac{d}{dr} \frac{e^{ikr}}{r} = 2 \mu_0 \omega k \sin\varphi \frac{e^{ikr}}{r}.$$

For the *directional characteristic* in the sense of p. 261 we obtain

$$(11) \qquad \sqrt{E} = M \sin\varphi,$$

where E is the radiated energy and M is the maximum of $\sqrt{E}$ that is radiated in the direction $\varphi = \pm \pi/2$ (M is proportional to r^{-1}).

We compare (11) with the elongated directional characteristic for the horizontal antenna in Fig. 29. The two curves are identical except for the interchange of $\sin\varphi$ and $\cos\varphi$, in accordance with the remark at the beginning of this section about the current in the frame and the horizontal antenna. The interchange of $\sin\varphi$ and $\cos\varphi$ is obviously due to the fact that while our horizontal antenna had the direction of the x-axis the plane of our frame antenna was situated perpendicular to the x-axis.

Hence, the frame antenna has its *maximal radiation in the plane of its frame* ($\varphi = \pm \pi/2$), just as the horizontal antenna has the maximal radiation in its own direction ($\varphi = 0$ and $\varphi = \pi$). Correspondingly, the frame antenna has *maximal reception* if its plane is situated *in the direction of the incoming wave.* Since this plane was assumed throughout to be the y,z-plane, the signal for maximal reception comes from the y-direction with dominating electric z-component (perpendicular to the ground) and magnetic x-component (perpendicular to the plane of the frame). Then the *electric z-component induces an electric current in the frame* or, as we may also put it, *the magnetic x-component stimulates the magnetic dipole of the frame.* Thus, the frame acts as a *magnetic receiver,* just as previously it acted as a *magnetic transmitter.*

Incidentally, in range finding we do not try for *maximal* reception but for *minimal* reception, which yields the more precise measurements, as in all zero methods of measuring in physics. The frame is then in the x,z-plane instead of the y,z-plane. The normal to the frame then points in the y-direction, i.e., *in the direction of the incoming signal.*

§ 36. Radiation Energy and Earth Absorption

In discussing certain energy questions we abandon the domain **E**, **H** of the field strengths that permit superposition, and turn to the quadratic quantity of energy flow

$$\mathbf{S} = [\mathbf{E}\,\mathbf{H}]$$

It now no longer suffices to consider the complex representation of the field under omission of the time factor $\exp(-i\,\omega\,t)$; instead we must multiply the real field components themselves. However, the complications which this brings with it can be eliminated by *averaging* over space and time. The mean values will be even simpler than our representation of the field so far, since due to the *orthogonality of the eigenfunctions,* the Bessel functions drop out of the representation and are replaced by more or less elementary functions.

Most important for our purposes is the total energy flow, integrated over a horizontal plane in the air:

$$S = \int \mathbf{S}_z \, d\sigma = \int (\mathbf{E}_r \, \mathbf{H}_\varphi - \mathbf{E}_\varphi \, \mathbf{H}_r) \, d\sigma. \tag{1}$$

Corresponding to whether this plane lies above ($z > h$) or below ($z < 0$) the dipole antenna ($z = 0$), we denote the energy flow (1) by S_+ or S_-. Both S_- and S_+ are taken relative to the positive z-direction. The energy that effectively[11] enters the earth in the *negative* z-direction is then given by $-S_-$, which, for the time being, is to be taken over the plane $z = 0$. However, we see that instead of this plane $z = 0$ we can use an arbitrary plane $z < h$, and in particular the planes $z = h - \varepsilon$, $\varepsilon \to 0$, for the computation of S_- (the space between two such planes is free from absorption and there is no noticeable energy loss in the direction of infinity). Since all energy which effectively enters the earth is transformed into Joule heat, the function $-S_-$ at the same time represents the total *thermal absorption of the earth* per unit of time. On the other hand S_+ taken over the planes $z = h + \varepsilon$, $\varepsilon \to 0$, measures the total radiation into the air above the plane $z = h$ per unit of time. We call S_+ the *effective radiation*. Hence

$$W = S_+ - S_- \tag{1a}$$

is the energy needed by the antenna per unit of time if we can neglect all energy losses in the antenna; or, in other words, it is the *power* needed by the antenna (the letter W reminds us of "watt"). In the following discussion we shall have to do mainly with this quantity W.

A. For the vertical antenna we had $\mathbf{E}_\varphi = 0$ and $\mathbf{H}_r = 0$. If we denote the expressions for $\mathbf{E}_r$ and $\mathbf{H}_\varphi$, which so far were complex, by E_r and H_φ and adjoin the time dependence, then (1) written explicitly becomes:

$$S = \frac{1}{4} \iint (E_r \, e^{-i\omega t} + E_r^* \, e^{+i\omega t}) \, (H_\varphi \, e^{-i\omega t} + H_\varphi^* \, e^{+i\omega t}) \, r \, dr \, d\varphi .$$

Upon averaging over time the terms involving $\exp(\pm 2 i \omega t)$ drop out, and, if from now on we understand S to be the mean value, we obtain

$$S = \frac{1}{4} \iint (E_r \, H_\varphi^* + E_r^* \, H_\varphi) \, r \, dr \, d\varphi .$$

[11] "Effective entry" means "excess of influx over outflux." The outgoing reflected radiation is of course automatically included in S_-.

Owing to the independence of the field from the φ- coordinate we can write this in the form

$$(2)\qquad S = \frac{\pi}{2}\int_0^\infty (E_r H_\varphi^* + E_r^* H_\varphi)\, r\, dr = \pi\, \mathrm{Re}\left\{\int_0^\infty E_r^* H_\varphi\, r\, dr\right\}.$$

For the computation of S_+ we take E_r and H_φ from (32.3) and for S_- we take them from (32.4). These expressions differ only by the signs of Π_{prim} in (32.3) and (32.4). We obtain

$$(3)\qquad E_r = \frac{\partial^2 \Pi}{\partial r\, \partial z} = \int_0^\infty I_1(\lambda r)\, f_1(\lambda, z)\, \lambda\, d\lambda.$$

$$(4)\qquad H_\varphi = \frac{-k^2}{\mu_0\, i\omega}\frac{\partial \Pi}{\partial r} = \frac{-k^2}{\mu_0\, i\omega}\int_0^\infty I_1(l r)\, f_2(l, z)\, l\, dl,$$

with

$$(5)\qquad f_1(\lambda, z) = \pm \lambda\, e^{-\mu|z-h|} + \mu\, F(\lambda)\, e^{-\mu(z+h)},$$

$$(6)\qquad f_2(l, z) = -\frac{l}{\mu_l}\, e^{-\mu_l|z-h|} - F(l)\, e^{-\mu_l(z+h)},$$

where $F(\lambda)$ and $F(l)$ are determined by (32.8). The fact that in (4) and (6) we used a variable of integration different from λ and hence had to replace μ by $\mu_l = \sqrt{l^2 - k^2}$, will prove useful in what follows. Using (3) and (4), equation (2) can be rewritten as follows:

$$(7)\qquad \frac{\mu_0\, \omega}{\pi k^2} S_\pm = \mathrm{Re}\left\{i \int_0^\infty f_1^*(\lambda, z)\, \lambda\, d\lambda \int_0^\infty f_2(l, z)\, l\, dl \int_0^\infty I_1(\lambda r)\, I_1(l r)\, r\, dr\right\}.$$

Here we can apply the *orthogonality relation* (21.9a), which we write in our present notation for the special case $n = 1$:

$$(8)\qquad \int_0^\infty I_1(\lambda r)\, I_1(l r)\, r\, dr = \delta(\lambda | l).$$

Hence, the right-most integral in (7) vanishes for all values of l except for $l = \lambda$, so that the middle integration in (7) yields $f_2(\lambda, z)$ (see the footnote on p. 111). Thus (7) reduces to the simple integral

$$(9)\qquad \frac{\mu_0\, \omega}{\pi k^2} S_\pm = \mathrm{Re}\left\{i \int_0^\infty f_1^*(\lambda, z)\, f_2(\lambda, z)\, \lambda\, d\lambda\right\}.$$

A further simplification is obtained if we let the planes $z = h \pm \varepsilon$

approach the position of the dipole antenna, that is

$$|z-h| = s \ll h, \quad z+h \sim 2h$$

Then instead of (5) and (6) we have

$$f_1(\lambda) = \pm \lambda e^{-\mu s} + \mu F(\lambda) e^{-2\mu h}, \tag{10}$$

$$f_2(\lambda) = -\frac{\lambda}{\mu} e^{-\mu s} - F(\lambda) e^{-2\mu h} \tag{11}$$

The product $f_1^*(\lambda) f_2(\lambda)$ in (9) is thus the sum of four terms. However, when we pass to the difference $S_+ - S_-$ only two terms remain, namely, those that correspond to the two signs of $f_1^*(\lambda)$ in (10). Applying the definition (1a) we obtain

$$\frac{\mu_0 \omega}{2\pi k^3} W = \text{Re}\left\{-i \int_0^\infty e^{-(\mu+\mu^*)s} \frac{\lambda^3 d\lambda}{\mu}\right\} + \text{Re}\left\{-i \int_0^\infty F(\lambda) e^{-2\mu h} \lambda^3 d\lambda\right\}, \tag{12}$$

where due to $s \ll h$ we may neglect $\mu^* s$ as compared to $2\mu h$ in the exponential function under the second integral sign. The first integral in (12) is easily evaluated. For $\lambda > k$ both μ and, of course, $\mu + \mu^*$ are real. Hence the real part of $-i$ times the integral from k to ∞ vanishes. Only the integral from 0 to k in which we may pass to the limit $s = 0$ remains. Using the variable of integration μ instead of λ we obtain[12]

$$\text{Re}\left\{-i \int_0^k \frac{\lambda^3 d\lambda}{\mu}\right\} = \text{Re}\left\{-i \int_{-ik}^0 (\mu^2 + k^2)\, d\mu\right\} = \frac{2}{3} k^3. \tag{13}$$

Concerning the second term in (12) we first consider the term $F(\lambda) = \lambda/\mu$ in (32.8) which does not vanish for $|k_E| \to \infty$, and thus compute:

$$\text{Re}\left\{-i \int_0^\infty e^{-2\mu h} \frac{\lambda^3 d\lambda}{\mu}\right\}. \tag{14}$$

Due to the real character of μ for $\lambda > k$ we again need consider only the integral from $\lambda = 0$ to $\lambda = k$. Written in terms of the variable μ, with the abbreviation $\zeta = 2kh$ (14) becomes

$$\text{Re}\left\{-i \int_{-ik}^0 e^{-\mu\zeta/k} (k^2 + \mu^2)\, d\mu\right\} = k^3\, \text{Re}\left\{\left(1 + \frac{d^2}{d\zeta^2}\right) \frac{e^{i\zeta} - 1}{i\zeta}\right\}.$$

[12] Due to the sign of μ we must follow the prescriptions concerning the "permissible sheet of the Riemann surface" in Fig. 28.

By evaluating the real part we obtain:

$$k^3\left(\frac{\sin\zeta}{\zeta}+\frac{d^2}{d\zeta^2}\frac{\sin\zeta}{\zeta}\right)=2\,k^3\,\frac{\sin\zeta-\zeta\cos\zeta}{\zeta^3}. \tag{15}$$

Combining (12), (13) and (15) we have

$$W=\frac{2\pi\,k^5}{\mu_0\,\omega}\left(\frac{2}{3}+2\,\frac{\sin\zeta-\zeta\cos\zeta}{\zeta^3}+K\right),\quad \zeta=2\,k\,h, \tag{16}$$

where K stands for the remaining contribution of $F(\lambda)$ for $|k_E|\neq\infty$, which was not yet considered in (14), namely,

$$K=\frac{1}{k^3}\ \mathrm{Re}\left\{i\int_0^\infty\frac{2\,\mu_E}{n^2\mu+\mu_E}\ e^{-2\mu h}\,\frac{\lambda^3\,d\lambda}{\mu}\right\}. \tag{17}$$

In connection with (16) we note that the first two terms on the right, which are independent of the nature of the ground, could have been deduced with the help of the apparatus of §31. However, the correction term K can be computed only with the help of our complete theory. We defer the discussion of these formulas to Section C.

B. For the *horizontal antenna* the formulas become more complicated due to the combined action of Π_x and Π_z, but with the help of the orthogonality relation (8) we finally obtain simplifications similar to those obtained for the vertical antenna. We shall merely outline the necessary computations. Instead of (2) we now have

$$S=\frac{1}{4}\ \mathrm{Re}\iint\left(E_r\,H_\varphi^*-E_\varphi\,H_r^*\right)r\,dr\,d\varphi \tag{2a}$$

and instead of (3) and (4) we have for the time being

$$E_r=k^2\cos\varphi\,\Pi_x+\frac{\partial}{\partial r}\,\mathrm{div}\,\vec{\Pi},\quad E_\varphi=-\,k^2\sin\varphi\,\Pi_x+\frac{1}{r}\frac{\partial}{\partial\varphi}\,\mathrm{div}\,\vec{\Pi}, \tag{3a}$$

$$H_\varphi=\frac{-k^2}{i\,\mu_0\,\omega}\left(-\cos\varphi\,\frac{\partial\Pi_x}{\partial z}+\frac{\partial\Pi_z}{\partial r}\right),\ H_r=\frac{-k^2}{i\,\mu_0\,\omega}\left(\sin\varphi\,\frac{\partial\Pi_x}{\partial z}+\frac{1}{r}\,\frac{\partial\Pi_z}{\partial\varphi}\right). \tag{4a}$$

Since, according to (34.1) and (33.15), $\mathrm{div}\,\vec{\Pi}$ and Π_z are proportional to $\cos\varphi$ while according to (33.12) Π_x is independent of φ, we conclude from (3a) and (4a) that E_r and H_φ contain the factor $\cos\varphi$, while E_φ and H_r contain the factor $\sin\varphi$. We then can carry out the integration with respect to φ in (2a), and instead of (7) we obtain a triple integral with respect to λ, l and r that has a somewhat complicated structure. However, if we form the difference $W=S_+-S_-$ then the formulas become much simpler, since only the term that arises from the primary stimulation Π_x has alternate signs. If we also use the Bessel differential equation for the elimination of the derivatives of I_0, then we obtain

$$\text{(5a)} \qquad \frac{W}{\pi k c} = \mathrm{Re}\left\{-i\int_0^\infty \lambda\, d\lambda\, A \int_0^\infty l\, dl\, e^{-\varepsilon\mu^*_l} \int_0^\infty r\, dr\, I_0(\lambda r)\, I_0(l r)\right\},$$

$$\text{(6a)} \qquad A = \frac{2k^2-\lambda^2}{\mu}\left(e^{-\varepsilon\mu} - e^{-2h\mu}\right) + 2\,\frac{\lambda^2 - 2\mu\mu_E}{n^2\mu+\mu_E}\, e^{-2h\mu}.$$

where ε is as before. Due to the orthogonality relation (8) this reduces to the simple integral:

$$\text{(7a)} \qquad \frac{W}{\pi k c} = \mathrm{Re}\left\{-i\int_0^\infty e^{-\varepsilon\mu^*} A\,\lambda\, d\lambda\right\}.$$

The integration of the term in (6a) that is independent of k_E can again be carried out as in (13) and (14), and again we need consider only the interval $0 < \lambda < k$. Thus, instead of (16) we obtain

$$\text{(16a)} \qquad W = \frac{2\pi k^5}{\mu_0\omega}\left(\frac{2}{3} - \frac{\sin\zeta}{\zeta} + \frac{\sin\zeta - \zeta\cos\zeta}{\zeta^3} + L\right)$$

where ζ is as before and

$$\text{(17a)} \qquad L = \frac{1}{k^3}\,\mathrm{Re}\left\{i\int_0^\infty e^{-2\mu h}\,\frac{2\mu\mu_E - \lambda^2}{n^2\mu+\mu_E}\,\lambda\, d\lambda\right\}.$$

The expression (16a) is free of Bessel functions, just as (16) is (see the beginning of this section). F. Renner has drawn my attention to the fact that the expressions (16) and (16a) can also be obtained by a process that may be more familiar to practical engineers, and that we shall discuss in exercise VI.3. However this process yields only the power $W = S_+ - S_-$ and not the values of S_+ and S_- separately, and the latter are of considerable practical interest.

C. *Discussion.* We first consider the principal terms of the equations (16) and (16a), neglecting for the time being the correction terms K and L:

$$\text{(18)} \qquad \begin{aligned} &\frac{2}{3} + 2\,\frac{\sin\zeta - \zeta\cos\zeta}{\zeta^3}, \\ &\frac{2}{3} - \frac{\sin\zeta}{\zeta} + \frac{\sin\zeta - \zeta\cos\zeta}{\zeta^3}. \end{aligned}$$

For $\zeta \to \infty$ they assume the common value 2/3. Due to $\zeta = 2kh$ the limit $\zeta = \infty$ is the same as $h = \infty$. Indeed, for $h = \infty$ the earth has no influence on the radiation of the antenna and the vertical and horizontal antennas must act in the same way. In both cases the total power is transformed into radiation. Correspondingly the equations (16) and (16a) yield the common limit

(18a)
$$W = \frac{4\pi}{3}\frac{k^5}{\mu_0\omega} = \frac{4\pi}{3}\frac{k^4}{\mu_0 c}.$$

This is identical with a formula given by Hertz[13] for the radiation of his dipole (freely oscillating in space). We note that the factor k^4 corresponds to the reciprocal fourth power of the wavelength in Rayleigh's law of scattering, which does actually arise from the superposition of a large number of distant dipoles that are distributed over the atmosphere and that are stimulated to radiation by the incoming sun rays.

If we expand the expressions (18) in ascending powers of ζ and then pass to the limit $h = 0$ so that the expansion breaks off with the term ζ^0, then we obtain:

(18b)
$$\frac{2}{3} + \frac{2}{3} + \cdots = \frac{4}{3} = 2\cdot\frac{2}{3} \text{ for the vertical antenna,}$$

$$\frac{2}{3} - 1 + \frac{1}{3}\cdots = 0\cdot\frac{2}{3} \text{ for the horizontal antenna.}$$

We can better understand the factors 2 and 0 on the right here with the help of Fig. 27 in §31: through reflection on an infinitely conductive earth the radiation of the vertical antenna doubles for $h = 0$, the radiation of the horizontal antenna is canceled by its mirror image. However we must remember that we have neglected the correction terms K and L in (18). This disregard of K and L means that simultaneously with passage to the limit $h \to 0$ we also let $k_E \to \infty$.

Figure 31 gives a general representation of the expressions (18). Above the axis of abscissas we have marked the values of ζ, below it the corresponding values of h. The figure shows that both for the vertical and the horizontal antenna the passage to the limit 2/3 is through continued oscillation around this limit. The distance between the abscissas of two consecutive extrema measured in the h scale is, for both curves, approximately equal to half a wavelength; this corresponds to the interference between the incoming radiation and the radiation which is reflected by the infinitely conductive ground.

In addition, for both curves we have traced a first correction by

[13] In the work quoted on p. 237; the formula can be found on p. 160 of his collected works v.II. In comparing (18a) with Hertz' formula we have to take into consideration the dimensionality factor which will be determined in (22) below.

broken lines as given by the terms K and L in (17) and (17a). The value $k/\mid k_E \mid = 1/100$ that we use corresponds to the case of sea water

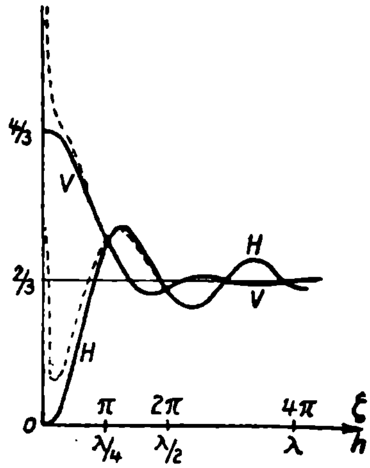


Fig. 31. The power needed by a dipole antenna for different altitudes h above the ground. V = vertical, H = horizontal antenna, the solid curves are for an infinitely conductive ground, the broken curves are for sea water.

for a 40-m. wavelength. We note that the ordinates of the broken curves increase steeply as h tends to zero; of course for finite h the difference in the ordinates from the limiting case $k_E \to \infty$ increases as k_E decreases. We are dealing here with a quite complicated double limit process, which reminds us of the double limit process in the Gibbs phenomenon of §2: if we first let $k_E = \infty$ and then $h \to 0$, we end up with the finite ordinates 4/3 and 0. However, if we stop at a finite value of k_E and first let $h \to 0$, then we end with an infinite ordinate which remains the same if afterwards we let $k_E \to \infty$.

What is the physical meaning of the infinite increase of W? *It does not add to the effective radiation S_+ but gets lost as earth heat* $- S_-$. In fact the Joule heat generated in the ground per unit of volume[14] for a fixed antenna current increases with increasing k_E, whereas the effective radiation remains finite. In order to prove this fact we should have to discuss the formulas for S separately, together with the correction terms K and L, and this would lead us too far afield.[15]

D. *Normalization to a given antenna current.* We have developed the entire theory of this chapter without consideration of the physical dimensions of the quantities introduced. This omission must be corrected now.

[14] Since the volume in which Joule heat is generated decreases with increasing $|k_E|$ (skin effect), we see that despite the statement in the text there is no heat loss in the limit $|k_E| \to \infty$.

[15] We refer the reader to the investigation by A. Sommerfeld and F. Renner, Strahlungsenergie und Erdabsorption bei Dipolantennen, *Ann. Physik* **41** (1942), where one also finds details concerning the concepts of radiation resistance and the form factor for a finite length of the antenna, which are customary in technology.

In the formula (31.1) we made the Hertz dipole factor equal to 1. In reality this factor is a denominate number, whose dimension is obtained from the relation between Π and $\mathbf{E}$ in (31.4). According to (31.4) Π has the dimension $\mathbf{E} \times M^2$. Since, according to (31.1), Π would have the dimension $1/r$, which is the same as M^{-1}, we obtain for the coefficient of Π, which we set equal to 1, the dimension $\mathbf{E} \times M^3$. We compare this factor with the Maxwell dielectrical translation $\mathbf{D} = \varepsilon\mathbf{E}$, which has the dimension of charge per unit of area, that is Q/M^2, where Q is the dimensional symbol for charge (see p. 237). Hence, written for the special case of the vacuum, we have the dimensional equation:

$$\varepsilon_0\, \mathbf{E} = \frac{Q}{M^2}, \qquad \text{hence} \qquad \mathbf{E}\, M^3 = \frac{QM}{\varepsilon_0}. \tag{19}$$

where QM is an electric momentum that we set equal to el. In Hertz' original model e was the charge of one particle which oscillated with respect to a resting charge $-e$ and beyond it.

Now what takes the place of this momentum in the case of the short antenna described on p. 237 that is loaded with end capacities? The current j_t that flows in the antenna must by assumption be constant over the whole antenna at every moment. We write it in the form:

$$j_t = j \sin \omega t = j \ \mathrm{Re}\{i\, e^{-i\omega t}\}. \tag{20}$$

The corresponding charges of the end capacities shall be

$$e_t = e \cos \omega t \quad \text{and} \quad = -e \cos \omega t.$$

According to the general relation

$$j_t = \frac{d}{dt} e_t$$

we must have $e = j/\omega$. At the time $t = 0$, when the current is zero, the charges of the end capacities are $\pm e$. Since these capacities are at the distance l of the length of the antenna they represent an electric momentum of magnitude

$$e\, l = \frac{j\, l}{\omega}. \tag{21}$$

We have to substitute this product el for the momentum QM in (19). In addition we have to append to (19) the factor $1/4\pi$ obtained from the comparison of the field (31.4) in the neighborhood of the dipole with the field of the antenna current. Thus we obtain for the dimensionality factor to be appended to our Π:

(22) $$\frac{jl}{4\pi\omega\varepsilon_0}.$$

Both the radiation S and the power W must be multiplied by the square of this factor. Using the relation

$$\frac{\omega}{k} = c = \frac{1}{\sqrt{\varepsilon_0\mu_0}}$$

we obtain instead of (16)

(23) $$W = \frac{1}{8\pi}k^2 l^2 \sqrt{\frac{\mu_0}{\varepsilon_0}}\, j^2\left(\frac{2}{3} + 2\,\frac{\sin\zeta - \zeta\cos\zeta}{\zeta^3} + K\right).$$

This formula gives the power in watts in a dimensionally consisten manner. Indeed, $\sqrt{\frac{\mu_0}{\varepsilon_0}}$ has the dimension of resistance and the numerical value $120\pi \doteq 377\,\Omega$. In our system, which is based on the unit of electricity $Q = 1$ coulomb, j is to be measured in amperes. Since kl has the dimension zero, W is expressed directly in units of power: $\Omega A^2 =$ watt.

After multiplication by the same factor, equation (16a) for the horizontal antenna becomes dimensionally correct; we find

(23a) $$W = \frac{1}{8\pi}k^2 l^2 \sqrt{\frac{\mu_0}{\varepsilon_0}}\, j^2\left(\frac{2}{3} - \frac{\sin\zeta}{\zeta} + \frac{\sin\zeta - \zeta\cos\zeta}{\zeta^3} + L\right).$$

Appendix

Radio Waves on the Spherical Earth

The earth will be assumed to be totally conductive (e.g., everywhere covered with sea water). We are dealing with a vertical antenna near the surface of the earth. The direction of the antenna is taken as the axis $\vartheta = 0$ of a polar coordinate system r, ϑ, φ; the distance of the antenna from the center of the earth is denoted by r_0; the radius of the earth by $a < r_0$. The field then consists of the components $\mathbf{E}_r, \mathbf{E}_\vartheta, \mathbf{H}_\varphi$ and is independent of φ. We now want to deduce this field from a scalar solution u of the wave equation.

The Hertz vector Π is not suitable for the representation of this field, since it satisfies not the simple wave equation $\Delta\Pi + k^2\Pi = 0$, but the more complicated form (31.3b) which holds for curvilinear coordinates. It is more convenient to start from the magnetic component $\mathbf{H}_\varphi = H e^{-i\omega t}$.

Using the second equation (31.5) we compute $\mathbf{E}_{r,\vartheta} = E_{r,\vartheta}\, e^{-i\omega t}$ from H

$$-i\,\omega\,\varepsilon_0\,E_r = \frac{1}{r\sin\vartheta}\frac{\partial}{\partial\vartheta}(\sin\vartheta H), \quad i\,\omega\,\varepsilon_0\,E_\vartheta = \frac{1}{r}\frac{\partial}{\partial r}(r\,H); \tag{1}$$

then, according to this scheme and from the φ-component of the first equation (31.5) we obtain

$$i\,\omega\,\mu_0\,r\,H = \frac{\partial}{\partial r}(r\,E_\vartheta) - \frac{\partial}{\partial\vartheta}E_r = \frac{1}{i\,\omega\,\varepsilon_0}\left\{\frac{\partial^2}{\partial r^2}(r\,H) + \frac{1}{r}\frac{\partial}{\partial\vartheta}\frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}(\sin\vartheta H)\right\}.$$

Hence H satisfies the differential equation

$$\frac{\partial^2}{\partial r^2}(r\,H) + \frac{1}{r}\frac{\partial}{\partial\vartheta}\frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}(\sin\vartheta H) + k^2\,r\,H = 0. \tag{2}$$

We can transform this equation into the wave equation $\Delta u + k^2 u = 0$ by making H proportional to $\partial u/\partial\vartheta$; for convenience we set in particular

$$H = i\,\omega\,\varepsilon_0\frac{\partial u}{\partial\vartheta}. \tag{2a}$$

Then (2) becomes

$$i\,\omega\,\varepsilon_0\,r\frac{\partial}{\partial\vartheta}\left\{\frac{1}{r}\frac{\partial^2 r\,u}{\partial r^2} + \frac{1}{r^2\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial u}{\partial\vartheta}\right) + k^2\,u\right\} = 0. \tag{3}$$

The first two terms in $\{\quad\}$ are equal to Δu according to the above mentioned scheme. Hence if we choose u as a solution of

$$\Delta u + k^2\,u = 0 \tag{4}$$

then according to (1) and (2a) the electromagnetic field is completely described by

$$E_r = -\frac{1}{r\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial u}{\partial\vartheta}\right), \qquad E_\vartheta = \frac{1}{r}\frac{\partial^2(r\,u)}{\partial\vartheta\,\partial r}, \qquad H = i\,\omega\,\varepsilon_0\frac{\partial u}{\partial\vartheta}. \tag{5}$$

The boundary condition $E_\vartheta = 0$ on the surface of the completely conductive earth is satisfied if we set

$$\frac{\partial r u}{\partial r} = 0 \quad \text{for} \quad r = a. \tag{5a}$$

In addition we have the condition that u is to behave as a unit source at the point $r = r_0$, $\vartheta = 0$, which means the existence of a radially directed dipole of the $\mathbf{E}$-field.

In this form the problem can be solved according to the method of §28 with the equation (22) of that section for $G(P,Q)$, the only difference being that then we had the boundary condition $u = 0$ instead of our present condition (5a). However, this implies only a change in the constant A. Whereas from (28.18) and the condition $u = 0$ we obtained the value (28.18a) for A, we now obtain from (5a)

$$A = -\left\{\frac{\partial}{\partial r} r\, \psi_n(k r) \Big/ \frac{\partial}{\partial r} r\, \zeta_n(k r)\right\}_{r=a}. \tag{6}$$

where here and in what follows ζ_n stands for ζ_n^1.

From the solution (28.22) modified in this way we first deduce the simplified formula for the limiting case $r_0 = a$, in which the antenna is directly on the ground. From (6) and (28.18) we obtain for this case

$$u_n(k r_0) = u_n(k a) = \psi_n(k a) - \frac{\psi_n(k a) + k a\, \psi_n'(k a)}{\zeta_n(k a) + k a\, \zeta_n'(k a)}\, \zeta_n(k a)$$

$$= k a \frac{\psi_n(k a)\, \zeta_n'(k a) - \zeta_n(k a)\, \psi_n'(k a)}{\zeta_n(k a) + k a\, \zeta_n'(k a)}.$$

According to exercise IV.8, equation II, the numerator of this fraction reduces to $i/(ka)^2$, so that if we abbreviate the denominator by $\xi_n(k a)$ we obtain

$$u_n(k a) = \frac{i/k a}{\xi_n(k a)}, \qquad \xi_n(k a) = \zeta_n(k a) + k a\, \zeta_n'(k a). \tag{6a}$$

Substituting this in the first line of (28.22) we obtain for G, which is our present u:

$$u = \frac{k}{4\pi i} \sum_{n=0}^{\infty} (2n+1)\, P_n(\cos\vartheta)\, \frac{\zeta_n(k r)}{\xi_n(k a)}, \tag{7}$$

This equation holds for all values $a < r < \infty$ and $0 \leq \vartheta \leq \pi$; the domain of validity of the lower line of the same equation (28.22) has now reduced to zero. This result (7) agrees with the previous treatment of this case by Frank-Mises (except for a factor which depends on our present definition of the unit source). In addition, the results there for arbitrary earth can be deduced here by a suitable extension of §28 (continuation into the interior of the sphere instead of the boundary condition on the surface).

If, further, we wished to treat the *horizontal* antenna on the spherical earth, then we should have to introduce, in addition to u, a function v which arises from the interchange of **H** and **E**, and we should obtain a representation for v that is similar to (7) but somewhat more complicated.[16]

However, the convergence of the series (7) is very poor, like that

[16] This was done by P. Debye in his dissertation, Munich 1908; *Ann. Physik* **30**, 67 (1909). See also Frank-Mises, Chapter XX, §4.

of Green's general representation in Chapter V. In order to see this for the present case we merely have to note that because of the ratio of earth radius to wavelength the numbers ka and kr are >1000. As long as n is of moderate magnitude, Hankel's asymptotic values for ζ are valid and they show that the ratio ζ_n/ξ_n in (7) is nearly independent of n. We should have to use more than 1000 terms of the series until the Debye asymptotic approximations (21.32) became valid; and only the latter can bring about a real convergence of the series.

In order to obtain a usable computation of u, we apply a method which was first applied successfully to our problem by G. N. Watson,[17] and which we shall find to be connected with the method developed in Appendix II to Chapter V. Namely, we transform the sum (7) into a *complex integral*.

To this end and on the basis of the relation

$$P_n(\cos\vartheta) = (-1)^n P_n(-\cos\vartheta).$$

which is valid for *integral* (and *only for integral*) n, we first rewrite the series in (7) in the form

$$\sum_{n=0}^{\infty}(2n+1)(-1)^n P_n(-\cos\vartheta)\frac{\zeta_n(kr)}{\xi_n(ka)} \tag{8}$$

We then replace n by a complex variable ν and we trace a loop **A** in the ν-plane of Fig. 32 that surrounds all the points

$$\nu = 0, 1, 2, 3, \ldots n, \ldots \tag{8a}$$

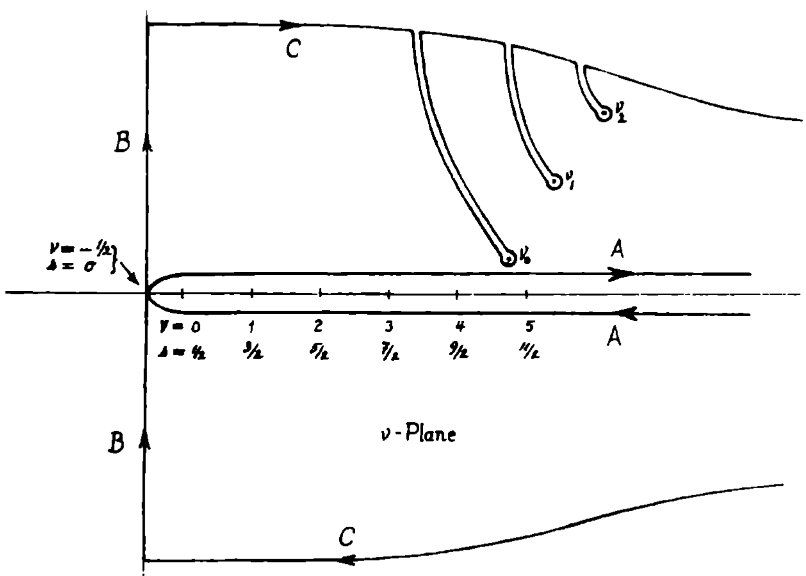


Fig. 32. Deformation of the loop **A** around the real axis. The curve **B** runs parallel to the imaginary axis of the ν-plane. The connection **C** between **B** and **A** must be considered situated at infinity.

[17] *Proc. Roy. Soc. London* **95** (1918).

in a clockwise direction. Over this loop we take the integral

$$\int \frac{2\nu+1}{2i\sin\nu\pi} P_\nu(-\cos\vartheta) \frac{\zeta_\nu(kr)}{\xi_\nu(ka)} d\nu, \tag{9}$$

which is obtained from the general term in (8) by interchanging n and ν, suppressing the factor $(-1)_n$, and appending the denominator $\sin\nu\pi$. As on p. 215, P_ν does not stand for the Legendre polynomial, but for the hypergeometric function

$$P_\nu(x) = F\left(-\nu, \nu+1, 1, \frac{1-x}{2}\right) \tag{9a}$$

(which is identical with the Legendre polynomial only for integral ν). Now the integrand of (9) has poles of first order at all the zeros of $\sin\nu\pi$; the zeros that lie inside the loop **A** are the points (8a) and in the neighborhood of the point $\nu = n$ we have as a first approximation:

$$\sin\nu\pi = \sin n\pi + (\nu - n)\pi\cos n\pi = (-1)^n \pi(\nu - n).$$

Hence the residue of the first fraction in (9) becomes

$$\frac{2n+1}{2i\pi}(-1)^n$$

and by computing the integral (9) as $-2\pi i$ times the sum of all residues we obtain

$$-\sum_{n=0}^{\infty}(2n+1)(-1)^n P_n(-\cos\vartheta)\frac{\zeta_n(kr)}{\xi_n(ka)}, \tag{10}$$

which is identical with (8) except for sign.

The next step consists in a deformation of the path **A**. We note that the hypergeometric series in (9a) is a *symmetric* function of its first two arguments. Hence we have for all (including complex) indices ν:

$$P_\nu = P_{-\nu-1}. \tag{11}$$

With the notation

$$\nu = s - \tfrac{1}{2} \tag{11a}$$

equation (11) becomes

$$P_{s-\frac{1}{2}} = P_{-s-\frac{1}{2}}. \tag{11b}$$

Hence $P_{s-\frac{1}{2}}$ is an *even* function of s.

This also holds for the last factor of the integrand in (9). In order to prove this we start from the representation (19.22) of H^1, which is valid for arbitrary indices; if we denote the index by s we have:

$$H_s^1(\varrho) = \frac{1}{\pi}\int_{W_1} e^{i\varrho\cos w} e^{is(w-\pi/2)} dw.$$

If we replace w by $-w$, s by $-s$ and reverse the orientation of W_1 we obtain:

$$H^1_{-s}(\varrho) = \frac{1}{\pi}\int_{W_1} e^{i\varrho\cos w}\, e^{is(w+\pi/2)}dw = e^{is\pi} H^1_{+s}(\varrho).$$

If we multiply this equation by $\sqrt{\pi/2\varrho}$ and use (21.15) in order to pass from H to ζ then we obtain

(11c) $$\zeta_{-s-\frac{1}{2}}(\varrho) = e^{is\pi}\, \zeta_{s-\frac{1}{2}}(\varrho).$$

The same relation holds for the quantity $\xi\,(k\,a)$ of (6a):

(11d) $$\xi_{-s-\frac{1}{2}}(k\,a) = e^{is\pi}\,\xi_{s-\frac{1}{2}}(k\,a).$$

By division we find that the quotient

(12) $$\frac{\zeta_{s-\frac{1}{2}}(kr)}{\xi_{s-\frac{1}{2}}(ka)}$$

is also an *even* function of s.

Finally the first fraction in the integrand of (9) written in terms of s is

(13) $$\frac{2s}{-2i\cos s\pi},$$

and therefore is an *odd* function of s.

We now deform the loop A into a straight line B (which is parallel to the imaginary axis of the ν- plane and passes through the point $s = 0$, i.e., $\nu = -\frac{1}{2}$) and two paths C (which are at a great distance from the real axis and, so to speak, join the ends of B with those of A). The poles that have to be considered in this connection will be discussed later. For the moment we show that the integrals over the paths B and C *vanish*.

For the path B this follows directly from the odd character of the integrand of (9) as written in terms of the variable s. In order to show the same thing for the path C we investigate the factor ζ_ν/ξ_ν of the integrand for large values of ν. We start from the series (19.34)

$$I_\nu(\varrho) = \frac{(\varrho/2)^\nu}{\Gamma(\nu+1)}(1 - \cdots),$$

where all the terms indicated by ... can be neglected for $|\nu| > \varrho$. According to Stirling's formula we have

$$\Gamma(\nu+1) = \sqrt{2\pi\nu}\, e^{-\nu}\nu^\nu,$$

hence

$$I_\nu(\varrho) = \frac{1}{\sqrt{2\pi\nu}}\left(\frac{e\varrho}{2\nu}\right)^\nu;$$

and for general complex ν:

$$I_{-\nu}(\varrho) = \frac{1}{\sqrt{-2\pi\nu}} \left(\frac{e\varrho}{-2\nu}\right)^{-\nu}. \tag{14}$$

From this follows:

$$\frac{I_\nu(\varrho)}{I_{-\nu}(\varrho)} = (-1)^{\nu+\frac{1}{2}} \left(\frac{e\varrho}{2\nu}\right)^{2\nu}.$$

This last quantity approaches zero if the real part of ν approaches plus infinity. Hence, in the representation (19.31) we can neglect I_ν as compared to $I_{-\nu}$. If from H_ν we pass to

$$\zeta_\nu = \sqrt{\frac{\pi}{2\varrho}}\, H_{\nu+\frac{1}{2}}$$

and from $\zeta_\nu(kr)$ we pass to the quotient of two ζ- functions, then, from (14) we obtain:

$$Z = \frac{\zeta_\nu(kr)}{\zeta_\nu(ka)} = \left(\frac{a}{r}\right)^{\nu+1}. \tag{15}$$

Since $a/r < 1$ the quantity in (15) vanishes if the real part of $\nu+1$ approaches plus infinity, as is the case on both parts of **C**. The same statement holds for the quotient $\zeta_\nu(kr)/\xi_\nu(ka)$, which according to (6a) and (14) can be written in the form

$$\xi_\nu(ka) = \zeta_\nu(ka)\left\{1 + \varrho\,\frac{\zeta'_\nu(\varrho)}{\zeta_\nu(\varrho)}\right\}_{\varrho = ka} = \zeta_\nu(ka)\,\{-\nu\}.$$

From this we see that the third factor of the integrand in (9) vanishes. The first factor vanishes due to the denominator $\sin\nu\pi$. The fact that the second factor vanishes follows from (24.17) which holds for an arbitrary complex index of the spherical harmonic. Hence our original path **A** can indeed be deformed through the infinite part of the half plane in which the real part of ν is positive.

However, in this deformation the path cannot cross the poles of the integrand:

$$\xi_\nu(ka) = 0. \quad \nu = \nu_0, \nu_1, \nu_2, \ldots \tag{15a}$$

We shall now investigate their position more closely. For the neighborhood of the m-th root we write:

$$\xi_\nu(ka) = (\nu - \nu_m)\,\eta_\nu(ka)\,, \qquad \eta_\nu = \left(\frac{\partial \xi_\nu}{\partial \nu}\right)_{\nu=\nu_m}. \tag{15b}$$

Then by forming residues we obtain from (9)

$$\pi \sum_{\nu=\nu_0,\nu_1,\nu_2,\ldots} \frac{2\nu+1}{\sin\nu\pi}\, P_\nu(-\cos\vartheta)\, \frac{\zeta_\nu(kr)}{\eta_\nu(ka)}. \tag{16}$$

Now, except for sign, the integral (9) is identical with the series (10) and the latter in turn is identical, except for a constant factor, with the solution (7) of the sphere problem. Hence the series (16) also represents the solution of the sphere problem, and suppressing the immaterial constant factor we may write:

$$u = \sum_{\nu} \frac{2\nu+1}{\sin \nu\pi} P_\nu(-\cos\vartheta) \cdot \frac{\zeta_\nu(kr)}{\eta_\nu(ka)}. \tag{17}$$

We see that the passage from the series (7), which is summed over integral n, to the series (17), which is summed over the complex ν, is obtained by *forming residues in a complex integral twice.*

Before we proceed with the discussion of (17) we return for a moment to Appendix II of Chapter V. There, too, we were dealing with a series summed over integral n and one summed over complex non-integral ν, namely, the series (1) and (3). We now show that there, too, the identity of the two series can be proved by forming residues in a complex integral twice. Written in analogy to (9) this integral is

$$\begin{aligned} &\int \frac{2\nu+1}{2i\sin\nu\pi} P_\nu(-\cos\vartheta)\, u_\nu(k, r_0)\, \zeta_\nu(kr)\, d\nu \qquad r > r_0, \\ &\int \frac{2\nu+1}{2i\sin\nu\pi} P_\nu(-\cos\vartheta)\, \zeta_\nu(kr_0)\, u_\nu(k, r)\, d\nu \qquad r < r_0, \end{aligned} \tag{18}$$

with
$$u_\nu(k,r) = \frac{\psi_\nu(kr)\,\zeta_\nu(ka) - \psi_\nu(ka)\,\zeta_\nu(kr)}{\zeta_\nu(ka)}.$$

We see that the poles of the integrands under consideration are the zeros of the denominator

$$\zeta_\nu(ka) = 0, \qquad \nu = \nu_1, \nu_2, \nu_3, \dots \nu_m \dots, \tag{18a}$$

which is common to the functions $u_\nu(k,r)$ and $u_\nu(k,r_0)$. The corresponding residues of $u_\nu(k,r)$ and $u_\nu(k,r_0)$ are

$$-\frac{\psi_\nu(ka)\,\zeta_\nu(kr)}{\eta_\nu(ka)} \quad \text{and} \quad -\frac{\psi_\nu(ka)\,\zeta_\nu(kr_0)}{\eta_\nu(ka)} \tag{18b}$$

where, in contrast to (15b), we have

$$\eta_\nu = \left(\frac{\partial\zeta_\nu}{\partial\nu}\right)_{\nu=\nu_m}. \tag{18c}$$

The original path of integration for the integrals in (18) is again the path A of Fig. 32. As in that figure, we can deform the path into the sum of the contours around the points $\nu = \nu_1, \nu_2, \dots$, since here too the paths B ,C make no contribution to the integral. Thus we obtain as the *common representation for the two integrals* (18)

$$(19) \qquad -\sum_{\nu} \frac{2\nu+1}{2i\sin\nu\pi} P_\nu(-\cos\vartheta) \frac{\psi_\nu(ka)}{\eta_\nu(ka)} \zeta_\nu(kr_0)\zeta_\nu(kr).$$

It can be seen that this coincides with the series (9), Appendix II to Chapter V, by considering the relation (II) from exercise (IV.8)

$$\zeta_\nu(\varrho)\psi_\nu'(\varrho) - \zeta_\nu'(\varrho)\psi_\nu(\varrho) = i/\varrho^2,$$

This relation yields the fact that for $\varrho = ka$ and $\zeta_\nu(ka) = 0$ the quantity $\psi_\nu(ka)$ is inversely proportional to $\zeta_\nu'(ka)$.

Thus the novel series of Appendix II can be derived by complex integration from ordinary series summed over integral n. In particular, this derivation shows the mathematical reason for the remarkable fact stressed on p. 217 paragraph 1, that the two n-series which are different for $r \gtrless r_0$ merge into the same ν-series (19).

In order to conclude our discussion of the spherical earth problem we must first show that the roots (15a) lie in the first quadrant of the ν-plane (just as the roots of (18a)). In (15a) we had the roots of the transcendental equation:

$$(20) \qquad \xi_\nu(ka) = (\zeta_\nu + \varrho\,\zeta_\nu')_{\varrho = ka} = 0.$$

For ζ_ν we again must use the special trigonometric form (11a) on p. 218 (both saddle points of equal altitude), since, for the general exponential form of ζ_ν, equation (20) can have no roots at all. Hence we can take $d\zeta_\nu/d\varrho$ from (11e), p. 219. We then obtain

$$(20\,a) \qquad \xi_\nu(ka) = \frac{i}{\varrho\sqrt{\sin\alpha}}(\sin z + \varrho\sin\alpha\cos z), \quad z = \varrho(\sin\alpha - \alpha\cos\alpha) - \frac{\pi}{4}.$$

Due to the quantity $\varrho = ka$ the second term in the parentheses (20a) is dominant. Hence, the roots of $\xi_\nu = 0$ (in contrast to the roots of $\zeta_\nu = 0$ on p. 219) are given with sufficient accuracy by

$$(20\,b) \qquad \cos z = 0, \qquad z = -\left(m+\frac{1}{2}\right)\pi \qquad \sin z = (-1)^m.$$

From this we obtain, in analogy to (21.40),

$$(21) \qquad \nu = \varrho\cos\alpha = ka\left(1-\frac{\alpha^2}{2}\right) = ka\left\{1+\frac{1}{2}(4m+1)^{\frac{2}{3}}\left(\frac{3\pi}{4ka}\right)^{\frac{2}{3}} e^{i\pi/3}\right\},$$

so that to each $m = 0,1,2,\ldots$ a root ν_m does indeed correspond in the first quadrant of the ν-plane.

Since the absolute values of the ν_m are large numbers (because $ka \gg 1$), we can compute $P_\nu(-\cos\vartheta)$ in (17) from the asymptotic equation (24.17). This equation can be written in the form

$$P_\nu(-\cos\vartheta) = \sqrt{\frac{1}{2\pi\nu\sin\vartheta}}\, e^{-i(\nu+\frac{1}{2})(\pi-\vartheta)+i\pi/4}.$$

if we neglect an exponentially small part. With the same accuracy we have

$$\sin\nu\pi = e^{-i\nu\pi}/2i.$$

Hence we can write in (17)

$$\frac{P_\nu(-\cos\vartheta)}{\sin\nu\pi} = \sqrt{\frac{2i}{\pi\nu\sin\vartheta}}\, e^{i(\nu+\frac{1}{2})\vartheta}. \tag{22}$$

We now specialize the factor ζ/η in (17) to the neighborhood of he surface of the earth, that is, we set $r = a$. According to equation (11a) on p. 218 with $\sin z = (-1)^m$ we now have

$$\zeta_\nu = \frac{i}{\varrho\sqrt{\sin\alpha}}(-1)^m$$

and if we restrict ourselves to the principal term of (20a) we obtain from (15b)

$$\eta_\nu = \frac{-i}{\varrho\sqrt{\sin\alpha}}\left(-\varrho\sin\alpha\,(-1)^m\right)\frac{\partial z}{\partial\nu}.$$

In analogy to p. 219 we have $\partial z/\partial\nu = -\alpha$. Hence,

$$\frac{\zeta_\nu}{\eta_\nu} = \frac{1}{\varrho\,\alpha\sin\alpha} = \frac{1}{ka\,\alpha\sin\alpha} \sim \frac{1}{ka\,\alpha^2}. \tag{23}$$

Substituting (22) and (23) in (17) we finally obtain

$$u = \frac{\sqrt{2i}}{ka}\sum_{\nu_m}\frac{2\nu+1}{\sqrt{\pi\nu\sin\vartheta}}\, e^{i(\nu+\frac{1}{2})\vartheta}\,\alpha^{-2}. \tag{24}$$

The last factor α^{-2} depends on the index of summation m; in fact we have as in (21)

$$\alpha^2 = (4m+1)^{\frac{2}{3}}\left(\frac{3\pi}{4ka}\right)^{\frac{2}{3}} e^{-2i\pi/3}.$$

In the first factor under the summation sign of (24) we may replace ν by the first term of the last member of (21), which is independent of m. Thus (24) simplifies to

$$u = \cdots (\sin\vartheta)^{-\frac{1}{2}} \sum_{\nu_m} (4m+1)^{-\frac{1}{3}} e^{i(\nu_m + \frac{1}{2})\vartheta}. \tag{25}$$

where the terms which are independent of m and ϑ are denoted by . . . . In our original series (7), summed over n, we would have had to consider more than 1000 terms. In our present series, summed over m, convergence is so rapid, due to the exponential dependence of the terms on $i\nu_m\vartheta$ and to the increase of ν_m indicated in (21), that we may break off at the first or second term. Because of the positive imaginary part of ν the increase of ν_m indicates an exponential damping of the radio signals with increasing distance along the surface of the earth; the factor $(\sin\vartheta)^{-\frac{1}{2}}$ indicates an increase of intensity at the antipodal point $\vartheta = \pi$ of the transmitter.[18]

We have omitted all numerical details since our formulas are of no importance for radio communication due to the predominant role of the ionosphere. However, our formulas are of interest for the general method of Green's function in Appendix II to Chapter V and they show the power of this method for a special example.[19]

[18] The apparent infinity of (22) for $\vartheta = \pi$ contradicts our general condition of continuity, but this need not disturb us since the equation (24.17) which was used in (22) loses its validity at the points $\vartheta = \pi$ and $\vartheta = 0$. A more precise investigation of the point $\vartheta = \pi$ leads to a kind of Poisson diffraction phenomenon of finite intensity (see J. Gratiatos, Dissertation, Munich; *Ann. Physik* **86** (1928)).

[19] I wish to mention the fact, communicated to me by Mr. Whipple on the occasion of a friendly visit by English physicists, that Watson's results can be deduced directly without the use of complex integration. I may venture the guess that the particular physical considerations which were made for this case are contained in the general method of our Appendix II to Chapter V.

EXERCISES FOR CHAPTER I

I.1. The position of the maxima and minima in special Fourier approximations. Show that the extrema of the Fourier approximation

$$S_{2n-1} = \sin x + \frac{1}{3} \sin 3x + \cdots + \frac{1}{2n-1} \sin (2n-1) x$$

lie at equal distances, and that except for the extremum $x = \pi/2$ which is common to all S, they lie between the extrema of S_{2n+1}.

I.2. Summation of certain arithmetic series. Compute the higher analogues to the Leibniz series (2.8) and to the series Σ_2, Σ_4 in (2.18).

I.3. Expansion of sin x in a cosine series. Expand $\sin x$ between 0 and π in a cosine series,

a) by considering $\sin x$ continued as an even function in the interval $-\pi < x < 0$, or

b) by substituting $b = 0$ and $c = \pi$ in (4.5).

I.4. Spectral resolutions of certain simple time processes. Compute the spectra of the time processes which are indicated in Fig. 33a and 33b from their Fourier integral and represent them graphically. In the same manner compute the spectrum of a sine wave $\sin 2\pi t/\tau$, which is bounded on both sides and which ranges from $t = -T$ to $t = +T$ where $T = n\tau$ (Fig. 33c), and deduce from this the fact that the width of a spectral line varies inversely with its life span. An absolutely sharp, completely monochromatic spectral line would therefore need a completely unperturbed sine wave that extends to infinity in both directions.

I.5. Examples of the method of complex integration. Give the reasons for the result of exercise I.4a (*Dirichlet discontinuous factor*) by the method of complex integration; also, resolve the spectrum of the sine curve bounded on both sides into the spectra of two waves that are bounded on one side.

I.6. Compute the first Hermite and Laguerre polynomials from their orthogonality condition by the method applied to spherical harmonics, normalizing so that the leading term of $H_n(x)$ is $(2x)^n$ and the constant term of $L_n(x)$ is $n!$.

For the definition of these polynomials see the table on p. 27.

EXERCISES FOR CHAPTER II

II.1. Elastic rod, open and covered pipe. Compute the *transversal* proper oscillations of a cylindrical rod of length l, which is clamped at $x = 0$ and oscillates freely at $x = l$, and compare them to the proper oscillations of an open and a covered pipe.

II.2. Second form of Green's theorem. Develop the analogue to Green's theorem, see v.II, equation (3.16), for the general elliptic differential expression $L(u)$,

a) where $L(u)$ is brought to the normal form,

b) in the general case.

c) Investigate the conditions under which the boundary value problem becomes unique for a self-adjoint differential expression L.

We may restrict ourselves to the case of two independent variables, for which Green's theorem is

$$\int u\,\Delta v\,d\sigma + \int (\operatorname{grad} u, \operatorname{grad} v)\,d\sigma = \int u\,\frac{\partial v}{\partial n}\,ds\,.$$

II.3. One-dimensional potential theory. Determine the one-dimensional Green's function from the conditions

a) $\dfrac{d^2G}{dx^2} = 0$ for $\begin{cases} 0 \le x < \xi\,, \\ \xi < x \le l\,, \end{cases}$

b) $G = 0$ for $x = 0$ and $x = l$,

c) $\dfrac{dG_+}{dx} - \dfrac{dG_-}{dx} = 1$ and G continuous for $x = \xi$,

and apply it to the (obviously trivial) solution of the boundary value problem:

$$\frac{d^2u}{dx^2} = 0\,, \quad u \text{ continuous for } 0 \le x \le l \begin{cases} u = u_0 & \text{for } x = 0, \\ u = u_l & \text{for } x = l. \end{cases}$$

Condition c) means "yield 1" of the source of G which is situated at $x = \xi$; G_+ is the branch $x > \xi$, G_- the branch $x < \xi$ of G.

II.4. Application of Green's method which was developed for heat conduction to the so-called laminar plate flow of an incompressible viscous fluid. We assume the flow to be planar and rectilinear throughout; this means that it is to be *independent* of the third coordinate z and *directed in the direction of the y-axis.* The velocity $\mathbf{v}$ then has the single component $\mathbf{v}_y = v$, which, due to our assumption of incompressibility, is independent not only of z, but also of y, so that the quadratic convection terms (**v** grad) **v** vanish. The Navier-Stokes equation for v is then according to v.II, equation (16.1)

$$\frac{\partial v}{\partial t} - k\frac{\partial^2 v}{\partial x^2} = -\frac{1}{\varrho}\frac{\partial p}{\partial y}; \tag{1}$$

where k is the kinematic viscosity; the right side is *independent of* x due to the corresponding equation for the vanishing x-component of velocity, hence it is a function of t only, say $a(t)$.

The flow is to be bounded at $x = 0$ by a fixed plate, which is at rest up to the time $t = 0$ and thereafter is in motion with the velocity $v_0(t)$. Due to the adhesion of the fluid to the plate we have for $x = 0$:

$$v = \begin{cases} 0 \ldots t \leq 0, \\ v_0(t) \ldots t > 0. \end{cases} \tag{2}$$

For the linear Couette flow (see v.II, Fig. 19b) we would have to add further boundary conditions on a plane that is at rest at a finite distance from $x = 0$. However, for the sake of simplicity, we shall consider this plate situated at infinity. The limiting case obtained in this manner is known in fluid dynamics as *plate flow*. For this flow we have, in addition to (2), the condition for $x = \infty$:

$$v = 0 \quad \text{and} \quad p = p_0 \text{ (independent of } y\text{)}. \tag{3}$$

From this it follows that $a(t) = 0$, so that (1) goes over into the equation of heat conduction.

We are thus led to a boundary value problem, which is a specialization of the problem illustrated by Fig. 13 only in that we now have $x_1 = \infty$ and $x_0 = 0$, and is a simplification of that problem because in the initial state in which the plate and hence the fluid are at rest, we have:

$$v = 0 \quad \text{for} \quad t = 0 \quad \text{and} \quad \text{all} \quad x > 0. \tag{4}$$

The solution is obtained as in (12.18), if the principal solution V is replaced by a suitable Green's function. Discuss the resulting velocity profile $v(x)$ for increasing values of t.

EXERCISES FOR CHAPTER III

III.1. Linear conductor with external heat conduction according to Fourier. Let the initial temperature for $x > 0$ be $u(x,0) = f(x)$. How must this function be continued for $x < 0$ so that the condition is

$$\frac{\partial u}{\partial n} + h\,u = 0$$

satisfied for $x = 0$?

III.2. Deduce *the normalization condition in anharmonic analysis* by specialization from Green's theorem.

III.3. Experimental determination of the ratio of outer and inner heat conductivity. A rod is to be kept at the fixed temperatures u_1 and u_3 at its ends $x = 0$ and $x = l$ and is to be in a *stationary* state after the effect of an arbitrary initial state dies out. The flow of temperature would then be *linear* if the lateral surface of the rod were adiabatically closed. Hence, at the middle section of the rod $x = l/2$ we would have temperature $u_2 = (u_1 + u_3)/2$. Hence from the measurement of

$$q = \frac{u_1 + u_3}{2\,u_2}$$

we can determine the ratio of outer and inner heat conduction (essentially our constant h). Deduce the relation between q and h needed for the evaluation of the measurement; according to the above $q = 1$ corresponds to $h = 0$.

III.4. Determination of the ratio of heat conductivity $\varkappa$ to electric conductivity σ. A metal rod is to be heated electrically, where the current i per unit of length gives the rod the Joule heat $i^2/q\sigma$ (q = cross-section of the rod); the rod is to be insulated against lateral heat conduction. Write the differential equation of the stationary state and adapt it to the boundary conditions $u = 0$ for $x = 0$ and $x = l$ that are realized in water baths. The potential difference V at the ends of the rod and the maximal temperature U at the mid-section of the rod are to be measured. From them we are to compute the ratio $\varkappa/\sigma$. For pure metals this ratio has a universal value (Wiedemann-Franz law).

EXERCISES FOR CHAPTER IV

IV.1. Power series expansion of $I_n(\varrho)$. Compute this expansion from the integral representation (19.18)

a) for integral n,

b) for arbitrary n

with the help of a general definition of the Γ-function.

IV.2. Deduce *the so-called circuit relations for H_n^1 and H_n^2* for integral n from the integral representations (19.22).

IV.3. Compute *the logarithmic singularity* of $H_0(\varrho)$ at the origin from the integral representations (19.22).

IV.4. An elementary process for the asymptotic approximation to $H_n^1(\varrho)$. Verify the asymptotic limiting value of $H_n^1(\varrho)$ by successively neglecting $1/\varrho$ and the higher powers of $1/\varrho$ already in the differential equation. This method is of course dubious from a mathematical point of view.

IV.5. Expansion of a function $f(\vartheta, \varphi)$ *on the sphere.*

a) Expand f first in a trigonometric series in φ, and then in spherical harmonics in $\cos\vartheta$. That is, find the expansions

$$f(\vartheta, \varphi) = \sum_{m=-\infty}^{+\infty} C_m e^{im\varphi}, \qquad C_m = \sum_{n=m}^{\infty} A_{mn} P_n^m(\cos\vartheta)$$

and as combination of both these expansions

$$(1) \qquad f(\vartheta, \varphi) = \sum_m \sum_n A_{mn} P_n^m(\cos\vartheta)\, e^{im\varphi} \begin{cases} -\infty < m < +\infty, \\ |m| \leq n < \infty. \end{cases}$$

b) Construct f from general surface spherical harmonics Y_n and determine the coefficients from the orthogonality relation for

$$Y_{nm} = P_n^m(\cos\vartheta)\, e^{-im\varphi},$$

that is, find

$$f(\vartheta, \varphi) = \sum_{n=0}^{\infty} Y_n, \qquad Y_n = \sum_{m=-n}^{+n} A_{nm} P_n^m(\cos\vartheta)\, e^{im\varphi};$$

and as combination of both expansions:

$$(2) \qquad f(\vartheta, \varphi) = \sum_n \sum_m A_{nm} P_n^m(\cos\vartheta)\, e^{im\varphi} \begin{cases} 0 < n < \infty, \\ -n \leq m \leq +n. \end{cases}$$

Clarify the apparent dissimilarity in the order of summation in (1) and (2) by a figure (lattice in the m,n-plane) and show that A_{mn} and A_{nm} in (1) and (2) formally have the same meaning (by interchanging the order of summation and integrating).

IV.6. Mapping of the wedge arrangement of Fig. 17 into circular crescents. Transform the 60°-angle wedge of Fig. 17 by reciprocal radii with a suitable position of the center of inversion C (see the text for this figure); the three straight lines 1,-1; 2,-2; 3,-3 then go into three circular arcs and the angles formed by them go into circular crescents. Examine the association of these regions and consider the fact that the Green's function of potential theory can be obtained for each of these regions (spatially speaking they are spherical crescents) by five repeated reflections.

IV.7. Mapping a) *of the plane parallel plate into two tangent spheres,* b) *of two concentric spheres into a plane and a sphere.* Investigate the three-dimensional figure into which the plane parallel plate of p. 74 together with its mirror images are transformed upon inversion. The sphere of inversion is best situated so that it is tangent to the boundary planes of the plate. Show that the plate is thus mapped into the exterior of two tangent spheres; and its mirror images are mapped into the space between two interior tangent spheres. b) Show that two concentric spheres can be inverted into a plane and a sphere. Hence, conversely, we can transform the potential of a sphere towards a plane into the much simpler boundary value problem for two concentric spheres. The same holds for the potential of two arbitrary non-intersecting spheres.

IV.8. Evaluation of two expressions involving Bessel functions. In equation (5) of Appendix I to Chapter IV determine

$$H_n(\varrho)\, I'_n(\varrho) - H'_n(\varrho)\, I_n(\varrho) \tag{I}$$

and in equation (20b) of the same appendix determine

$$\zeta_n(\varrho)\, \psi'_n(\varrho) - \zeta'_n(\varrho)\, \psi_n(\varrho). \tag{II}$$

EXERCISES FOR CHAPTER V

V.1. Normalization questions. Normalize the functions $I_n(\lambda r)$ and $\psi_n(k r)$ of (26.3) and (26.2) to 1 for the basic interval $0 < r < a$ with the boundary conditions $I'_n(\lambda a) = 0$ and $\psi'_n(k a) = 0$ in analogy to equation (20.19).

V.2. The Gauss theorem of arithmetic mean in potential theory. Prove the theorem: The value of a potential function $\overline{u}$ at any point P of its domain of regularity S is equal to the arithmetic mean u of its values on an arbitrary sphere K_a, which has radius a and center P and which lies entirely in S.

V.3. Summation formulas over the roots of Bessel functions. Verify that the coefficients A_{nm} in the expansion (27.13) equal those of (27.14), and derive interesting identities for the Ψ_n from the equality of the coefficients of $\Pi_n^m(\cos\vartheta_0)\, e^{-im\varphi_0}$ in these two expansions. These identities can be rowritten as identities for the ψ_n.

EXERCISES FOR CHAPTER VI

VI.1. Vertical and horizontal antenna at the altitude h over an infinitely conductive ground. Show that the formulas (31.16) and (31.17) for the two *electric* antennas:

$$\text{a)}\quad \Pi = \Pi_z = \frac{e^{ikR}}{R} + \frac{e^{ikR'}}{R'}, \qquad \text{b)}\quad \Pi = \Pi_x = \frac{e^{ikR}}{R} - \frac{e^{ikR'}}{R'},$$

satisfy the conditions (31.15) for the vanishing of the tangential component of **E** in the entire plane $z = 0$.

In the same manner show that the formulas (31.19) and (31.20) for the two *magnetic* antennas:

$$\text{c)}\quad \Pi = \Pi_z = \frac{e^{ikR}}{R} - \frac{e^{ikR'}}{R'}, \qquad \text{d)}\quad \Pi = \Pi_x = \frac{e^{ikR}}{R} + \frac{e^{ikR'}}{R'}$$

satisfy the conditions (35.1) for the vanishing of the tangential electric component.

VI.2. Behavior of the electric force lines for a Zenneck wave in the neighborhood of the earth's surface. Show that the lines of force in the air are bent forward, i.e., in the direction of propagation and that the lines of force in the earth are *dragged behind.*

VI.3. Simplified computation of the power needed for the vertical and horizontal antenna. Prove the expressions (36.16) and (36.16a) by determining the work $\mathbf{E}jl$ done per unit of time by the field strength **E** and the current j for the length l of the antenna.

HINTS FOR SOLVING THE EXERCISES

I.1. The equation which determines the position of the extrema is

$$\cos x + \cos 3x + \cdots + \cos(2n-1)x = 0. \tag{1}$$

If we write this in the form

$$\operatorname{Re}(e^{ix} + e^{3ix} + \cdots + e^{(2n-1)ix}) = 0, \tag{2}$$

then we can sum this geometric series. We obtain finally

(3) $$\frac{\sin 2nx}{2\sin x} = 0, \qquad \text{hence} \qquad (4)\ \ x = \frac{\pi}{2n},\ \frac{2\pi}{2n}, \ldots, \frac{(2n-1)\pi}{2n}.$$

Due to the denominator in (3) we do not count the points $x = 0$ and $x = \pi$. Equation (4) proves the statement in the exercise.

I.2. By integrating from 0 to x we obtain a sine series from the cosine series (2.17); and if in the sine series we set $x = \pi/2$ then we obtained the analogue to the Leibniz series which follows (2.14). Integrating this sine series we obtain a series in terms of $1 - \cos x$, $1 - \cos 3x, \ldots$ and setting $x = \pi/2$ here we obtain the next analogue to (2.16), from which we deduce the value of Σ_6. This process can be continued indefinitely, but it does not seem to lead to a transparent law for the successive analogues.

I.3. The two processes mentioned in the exercise lead to the series

$$\sin x = \frac{2}{\pi}\left(1 - \frac{2}{1\cdot 3}\cos 2x - \frac{2}{3\cdot 5}\cos 4x \ldots\right)$$

from which, by setting, e.g., $x = 0$, we obtain a representation of ½ as a series in the reciprocals of odd integers.

I.4. Consider case a). If in (4.8) we replace x by t and perform the integration with respect to ξ we obtain

$$f(t) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{d\omega}{\omega}\sin\frac{\omega\tau}{2}\, e^{i\omega t} = \int_0^{\infty} a(\omega)\cos\omega t\, d\omega. \tag{1}$$

where $|a(\omega)|$ stands for the amplitude of the spectrum of $f(t)$ with the frequency ω and, according to (1),

$$a(\omega) = \frac{1}{\pi}\frac{\sin\omega\tau/2}{\omega/2}. \tag{2}$$

This function has its principal maximum of altitude τ/π at $\omega = 0$.

followed by secondary maxima of decreasing altitudes at intervals with lengths asymptotically equal to $\Delta\omega = 2\pi/\tau$.

In case b), where according to the figure we are dealing with a function f which is odd in t, we obtain in the same manner:

$$(3)\quad \begin{aligned} f(t) &= \frac{i}{\pi}\int_{-\infty}^{+\infty} \frac{d\omega}{\omega}\left(1-\cos\frac{\omega\tau}{2}\right)e^{i\omega t} = \int_0^{\infty} b(\omega)\sin\omega t\, d\omega; \\ b(\omega) &= -\frac{1}{\pi}\frac{1-\cos\omega\tau/2}{\omega/2} = -\frac{1}{\pi}\frac{\sin^2\omega\tau/4}{\omega/4}. \end{aligned}$$

We now have $b(\omega) = 0$ for $\omega = 0$; the first maximum lies at $\omega \sim 4.7/\tau$; as before, the subsequent secondary maxima successively decrease in altitude.

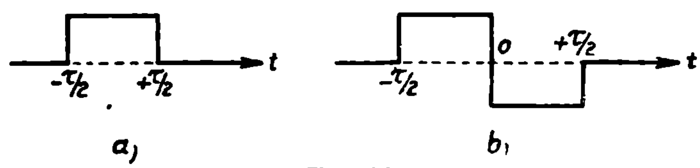


Fig. 33.

a) $f(t) = 0$ for $|t| > \tau/2$,
$f(t) = 1$ for $|t| < \tau/2$.

b) $f(t) = 0$ for $|t| > \tau/2$,
$f(t) = 1$ for $-\tau/2 < t < 0$,
$= -1$ for $0 < t < \tau/2$.

In both cases a) and b) we are dealing with a "grooved spectrum" that extends to infinity.

In the beginning of the theory of x-rays an attempt was made to interpret them as ether impulses of the type a) or b). From a spectral point of view, which is the only one that is physically justified, this is not a departure from the wave interpretation, but merely an (arbitrary special) assumption about the nature of the x-ray spectrum.

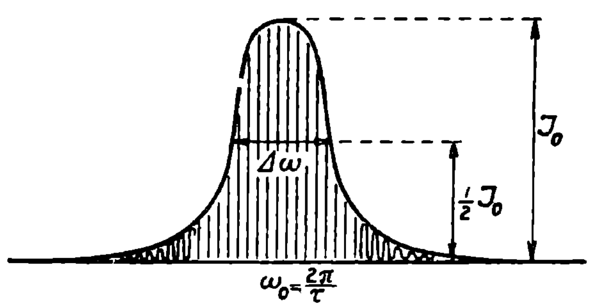


Fig. 33c. Schematic representation of the intensity in the spectrum of a wave process of frequency $\omega_0 = \frac{2\pi}{\tau}$, which breaks off on both sides. Here $\Delta\omega$ is the half-value width of the corresponding spectral line. The ruled middle portion of the spectrum consists of $\sin^2$-like oscillations, just like the unruled part.

For the wave process of length $2T = 2n\tau$ which breaks off on both sides (Fig. 33c) we start most conveniently from equation (4.11b) and find

$$(4) \qquad f(t) = \int_0^\infty b(\omega) \sin \omega t \, d\omega, \qquad b(\omega) = \frac{4}{\tau} \frac{\sin \omega T}{\omega^2 - \left(\frac{2\pi}{\tau}\right)^2}.$$

The principal maximum of b lies, as expected, at the frequency $\omega_0 = 2\pi/\tau$ and has the altitude $b_0 = n\tau/\pi$ corresponding to the intensity $I_0 = \left(\frac{n\tau}{\pi}\right)^2$, and hence increases with increasing length. This principal maximum is flanked on both sides by secondary maxima of successively decreasing altitudes at intervals with length asymptotically approaching $\pi/n\tau$; for all these maxima we have $\sin \omega T \sim 1$. We seek the two maxima for which $I = \frac{1}{2}I_0$, that is, according to (4), those maxima for which

$$(5) \qquad \omega^2 - \omega_0^2 = \pm \frac{4\sqrt{2}\,\pi}{n\tau^2} = \pm \frac{\sqrt{2}}{\pi n} \omega_0^2, \quad \omega^2 = \omega_0^2 \left(1 \pm \frac{\sqrt{2}}{\pi n}\right).$$

The difference of their frequencies is the so-called half-value width of the corresponding spectral line. If we assume $n \gg 1$ then this frequency difference is, according to (5),

$$(6) \qquad \Delta\omega = \frac{\sqrt{2}}{\pi n} \omega_0 = \frac{2\sqrt{2}}{T}.$$

Hence the half value width decreases with increasing T as stated in the exercise. Only for $T \to \infty$ do we obtain an absolutely sharp spectral line.

I.5. The function $f(t)$ of Fig. 33a) for $\tau/2 = 1$ is known in the mathematical literature as the *Dirichlet discontinuous factor:*

$$(1) \qquad D = \frac{2}{\pi} \int_0^\infty \sin \omega \cos \omega t \frac{d\omega}{\omega} = \begin{cases} 1 & |t| < 1, \\ \frac{1}{2} & |t| = 1, \\ 0 & |t| > 1. \end{cases}$$

If we set $t = 0$ here then we obtain the fundamental integral

$$(2) \qquad \int_0^\infty \sin \omega \frac{d\omega}{\omega} = \frac{\pi}{2} \qquad \text{or} \qquad (2a) \quad \int_{-\infty}^{+\infty} \sin \omega \frac{d\omega}{\omega} = \pi.$$

which was used in connection with Fig. 4, p. 11. This can be verified directly through complex integration: since $\sin \omega/\omega$. is analytic on the real axis and in its neighborhood, we can avoid the point $\omega = 0$ (e.g., as in Fig. 34a, p. 301) below the real axis, and we then can decompose (2a) into the difference of the integrals

$$\text{(3)} \qquad \mathrm{I} = \frac{1}{2i}\int e^{i\omega}\frac{d\omega}{\omega}, \qquad \mathrm{II} = \frac{1}{2i}\int e^{-i\omega}\frac{d\omega}{\omega},$$

both integrals being taken over the heavy line of the figure. The path of integration in II can be deformed into the infinite part of the *lower* half plane, where the integral vanishes. The path of integration of I must be deformed into the infinite part of the upper half plane since $\exp(i\omega)$ vanishes only there; however, it cannot be deformed across the pole $\omega = 0$. The residue at the pole is $2\pi i$. This proves (2a) and hence (2).

Finally, we easily verify that those parts of the path of integration which in the figure are indicated by short arrows and their dotted continuations also make no contribution to I and II.

The method of complex integration also serves to extend the statements of the preceding exercise for the wave which is bounded on both sides. Such a wave can be considered as the superposition of two waves, which are bounded on one side, of opposite phase, one ranging from $t = -T$ to $t = \infty$, the other from $t = +T$ to $t = \infty$. However these processes cannot be represented individually in the Fourier manner, due to the divergences at $t = \infty$. For this purpose it is necessary to transfer the path of integration in equation (4) of the preceding exercise from the real axis into the complex domain (as shown in Fig. 34b), and then to perform the decomposition. This is explained by the following transformations which start from equation (4) on p. 299:

$$f(t) = \frac{4}{\tau}\int_0^\infty \sin\omega t\,\sin\omega T\,\frac{d\omega}{\omega^2 - \left(\frac{2\pi}{\tau}\right)^2}$$

$$= \frac{2}{\tau}\int_0^\infty \left\{\cos\omega(t-T) - \cos\omega(t+T)\right\}\frac{d\omega}{\omega^2 - \left(\frac{2\pi}{\tau}\right)^2}$$

$$= \frac{1}{\tau}\int_{-\infty}^{+\infty} \left(e^{i\omega(t-T)} - e^{i\omega(t+T)}\right)\frac{d\omega}{\omega^2 - \left(\frac{2\pi}{\tau}\right)^2} = \mathrm{I}-\mathrm{II},$$

$$\left.\begin{matrix}\mathrm{I}\\ \mathrm{II}\end{matrix}\right\} = -\frac{1}{\tau}\int e^{i\omega(t\pm T)}\frac{d\omega}{\omega^2 - \left(\frac{2\pi}{\tau}\right)^2},$$

where the integral signs without upper and lower limits are to be taken over the complex path of Fig. 34b.

We claim that I represents the wave process starting at $t = -T$

and II represents that starting at $t = +T$, both continuing to $t = \infty$.

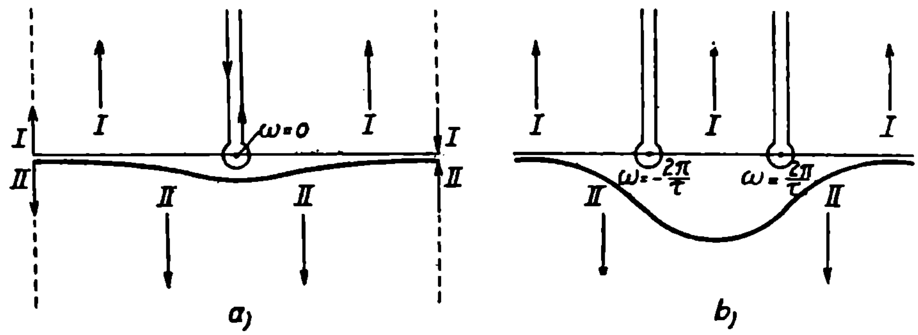


Fig. 34. a) Dirichlet discontinuous factor. We prove equation (3) by deforming the path of integration of II downward, that of I upward. b) Wave process which breaks off on both sides. The complex integrals I and II represent wave processes which are bounded on one side.

In order to prove this we set $T = 0$ for simplicity, and show that

$$f_0(t) = -\frac{1}{\tau}\int e^{i\omega t}\frac{d\omega}{\omega^2 - \left(\frac{2\pi}{\tau}\right)^2} \tag{4}$$

is a sine wave which starts at $t = 0$ and continues to $t = \infty$. The proof is given in a manner similar to that of (3): for $t < 0$ the path of integration can be drawn into the infinite part of the lower half plane and the integral vanishes there. For $t > 0$ the path of integration must be drawn into the upper half plane. Due to the poles $\omega = \pm 2\pi/\tau$ we obtain the residues

$$2\pi i\,\frac{e^{2\pi i t/\tau}}{4\pi/\tau} \qquad \text{and} \qquad 2\pi i\,\frac{e^{-2\pi i t/\tau}}{-4\pi/\tau};$$

so that

$$f_0(t) = -\frac{i}{2}\left(e^{2\pi i t/\tau} - e^{-2\pi i t/\tau}\right) = \sin\frac{2\pi t}{\tau}.$$

This completes the proof.

If instead of starting from (4) we start from

$$f_0(t) = -\frac{1}{2\pi}\,\mathrm{Re}\int e^{i\omega t}\frac{d\omega}{\omega - 2\pi/\tau} \tag{5}$$

then we see that for the same choice of the path of integration and the same deformation of this path we obtain the same result as before:

$$f_0(t) = \begin{cases} 0 & t < 0, \\ \sin\dfrac{2\pi t}{\tau} & t > 0. \end{cases} \tag{6}$$

The interest in this representation lies in the optic theory of dispersion. We imagine that perpendicularly to the plane $x = 0$ the wave (6) enters a medium filling the half space $x > 0$, and decompose it according to (5) into partial waves of the form $a(\omega)e^{i\omega t}$. Each of these waves propagates in the direction of increasing x independently of all other waves and we represent it here by $a(\omega)\exp[i(kz-\omega t)]$. The wave number k in a dispersion-free medium would be $k_0 = \omega/c$; due to the induced oscillation of the electrons (numbering N per cm.[3]) we have

$$k^2 = \frac{\omega^2}{c^2}\left(1 - \frac{Ne^2/m}{\omega^2-\omega_e^2}\right), \tag{7}$$

where ω_e is the proper frequency of the oscillating electrons (for the sake of simplicity we neglect the damping of these oscillations). Hence in the infinite part of the ω-plane we have:

$$k = k_0, \qquad kx - \omega t = k_0 x - \omega t = \frac{\omega}{c}(x - ct).$$

Thus the question of whether the path of integration is to be deformed in the direction of the positive or negative half plane is determined by the sign of $x - ct$. Since this criterion is independent of ω it is the same for all partial waves, so that $x = ct$ stands for the entire light stimulation at the point x of the dispersive medium. *The head of our light signal therefore propagates with the vacuum velocity* $dx/dt = c$, *not*, as one might think, *with the phase velocity* $V = \omega/k$ *which is characteristic for the dispersive medium.* We interpret this in the following way: the electrons are at rest for $t < x/c$ and start plane oscillation for $t = x/c$. The full amplitude corresponding to the incoming oscillation is attained only at a later time that is determined not by the phase velocity V but by the *group velocity* $U = d\omega/dk$. The oscillation processes which precede this time may be called *forerunners of the light signal.*

I.6. a) *Hermite polynomials.* Due to the even character of $g(x) = e^{-x^2}$ and the fact that the interval is $-\infty < x < +\infty$ we see that the Hermite polynomials, like the spherical harmonics P_n, are even or odd functions of x depending on whether n is even or odd. Considering this fact and the customary normalization of H_n (see exercise) write:

$$H_0 = 1, \quad H_1 = 2x, \quad H_2 = 4x^2 + a, \quad H_3 = 8x^3 + bx, \quad H_4 = 16x^4 + cx^2 + d$$

and compute the coefficients a, b, c, d through a repeated application of the orthogonality condition (result:

$$a = -2,\ b = -12,\ c = -48,\ d = +12).$$

b) *Laguerre polynomials.* Due to the weighting factor $g(x) = e^{-x}$ and the fact that the interval is $0 < x < \infty$ the polynomials are no longer even or odd. Considering this fact and the customary normalization (see exercise) write:

$L_0 = 1, \quad L_1 = a x + 1, \quad L_2 = b x^2 + c x + 2, \quad L_3 = dx^3 + e x^2 + fx + 6$

and compute the coefficients $a, b, \ldots f$ as in a) result: $a = -1$, $b = 1$, $c = -4$, $d = -1$, $e = 9$, $f = -18$).

II.1. The differential equation (7.8) to which this exercise relates is obtained as follows from the theory of *beam bending:* we start from the differential equation:

$$\frac{\partial^4 u}{\partial x^4} = \frac{1}{EI} \frac{\partial^2 M}{\partial x^2}. \tag{1}$$

The bending moment M of the exterior static load of the beam is to be replaced by the moment of the dynamic inertia resistances

$$-\varrho \frac{\partial^2 u}{\partial t^2}$$

(ϱ = mass per unit of length of the oscillating beam). Let this beam be clamped at $x = 0$ and let the free end be $x = l$. All the cross-sections $x < \xi < l$ contribute to the bending moment at the cross-section x, each cross-section with the lever-arm $\xi - x$. Hence we have

$$M = -\varrho \int_x^l (\xi - x) \frac{\partial^2 u}{\partial t^2} d\xi, \quad \frac{\partial M}{\partial x} = \varrho \int_x^l \frac{\partial^2 u}{\partial t^2} d\xi, \quad \frac{\partial^2 M}{\partial x^2} = -\varrho \frac{\partial^2 u}{\partial t^2}. \tag{2}$$

Substituting this in (1) we obtain (7.8) and for the constant c (of dimension cm.^{2}/sec) we obtain

$$c = \sqrt{\frac{EI}{\varrho}}. \tag{3}$$

According to (2) we have at the free end $M = \frac{\partial M}{\partial x} = 0$. According to equation (1) above this means

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^3 u}{\partial x^3} = 0 \qquad x = l. \tag{4}$$

On the other hand the clamping implies

$$u = \frac{\partial u}{\partial x} = 0 \quad \text{for } x = 0. \tag{5}$$

If we write $u = U e^{i\omega t}$, $U = e^{\alpha x}$ then (7.8) yields

$$\alpha^4 = \frac{\omega^2}{c^2}, \qquad \alpha = \pm k \text{ and } = \pm i k, \qquad k = \sqrt{\frac{\omega}{c}}.$$

Hence there are four particular solutions of the differential equation for U:

$$e^{kx}, \; e^{-kx}, \; e^{ikx}, \; e^{-ikx},$$

which for the following can be combined more conveniently into the forms

$$\sinh kx, \; \cosh kx, \; \sin kx, \; \cos kx.$$

Hence the general solution becomes

$$U = A \sinh kx + B \cosh kx + C \sin kx + D \cos kx.$$

According to (4) and (5) there are four relations among the constants of integration A,B,C,D, from which we obtain through elimination the transcendental equation

$$\text{(6)} \qquad \cos kl = -\frac{1}{\cosh kl}.$$

The graphical treatment of this equation in the manner of Fig. 7 yields an infinity of roots at intervals which asymptotically become

$$\text{(7)} \qquad k_{n+1} - k_n = \frac{\pi}{l}, \qquad \omega_{n+1} - \omega_n = 2\,c\,n\,\frac{\pi^2}{l^2}$$

For the basic oscillation we have

$$\text{(7a)} \qquad k = k_1 = 1.875/l, \qquad \omega = \omega_1 = c\,k_1^2.$$

The differential equation of a *pipe* is the same as that of an oscillating string, that is equation (7.6) where u = longitudinal velocity of air, c = velocity of sound. For the pipe which is open on both ends, or one which is covered at $x = 0$ and open at $x = l$ we have the boundary conditions

$$\text{(8a)} \qquad \frac{\partial u}{\partial x} = 0 \quad \text{for } x = 0 \quad \text{and} \quad x = l \text{ (open pipe)}$$

or

$$\text{(8b)} \qquad u = 0 \quad \text{for} \quad x = 0, \quad \frac{\partial u}{\partial x} = 0 \quad \text{for} \quad x = l \text{ (covered pipe)}$$

(due to the hydrodynamic continuity equation $\partial u/\partial x = 0$ means the same as $\partial p/\partial t = 0$, that is $p = p_0$ = atmospheric pressure which we assume to hold approximately). Writing $u = U\,e^{i\omega t}$ for the proper oscillations we obtain from (8a, b)

$$\text{(9a)} \qquad U = A \cos k_n x, \quad k_n = n\,\frac{\pi}{l}, \qquad \omega_n = c\,k_n, \quad k_1 = 3.14/l$$

$$\text{(9b)} \qquad U = B \sin k_n x, \quad k_n = \left(n + \frac{1}{2}\right)\frac{\pi}{l}, \quad \omega_n = c\,k_n, \quad k_1 = 1.57/l.$$

The value (7a) of k_1 lies between the values (9b) and (9a). The sequence of ω is harmonic for both the open and covered pipe; for the elastic rod it becomes harmonic only asymptotically for high overtones (see (7)).

II.2. a) Using the identities

$$v\frac{\partial^2 u}{\partial x^2}+\frac{\partial v}{\partial x}\frac{\partial u}{\partial x}=\frac{\partial}{\partial x}\left(v\frac{\partial u}{\partial x}\right),$$

$$D v\frac{\partial u}{\partial x}+u\frac{\partial D v}{\partial x}=\frac{\partial}{\partial x}(D u v),$$

and writing $L(u)$ in the normal form (10.1), we obtain

$$v L(u)+\Lambda(u,v,\ldots)=\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y} \tag{1}$$

with

$$\Lambda(u,v,\ldots)=(\operatorname{grad} u,\ \operatorname{grad} v)+u\left(\frac{\partial D v}{\partial x}+\frac{\partial E v}{\partial y}-F v\right), \tag{2}$$

$$X=v\frac{\partial u}{\partial x}+D u v,\qquad Y=v\frac{\partial u}{\partial y}+E u v. \tag{3}$$

Here Λ is a *bilinear form* in the u,v and their first derivatives. If L is self-adjoint then, because $D=E=0$, equation (10.6), we have:

$$\Lambda=(\operatorname{grad} u,\ \operatorname{grad} v)-F u v, \tag{4}$$

which is symmetric in u and v.

b) If L has the general form (8.1) then we again have equation (1), but with

$$\Lambda(u,v\ldots)=A\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}+B\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}+\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right)+C\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}$$
$$+u\left(\frac{\partial D v}{\partial x}+\frac{\partial E v}{\partial y}-F v\right)+v\left(\frac{\partial A}{\partial x}\frac{\partial u}{\partial x}+\frac{\partial B}{\partial x}\frac{\partial u}{\partial y}+\frac{\partial B}{\partial y}\frac{\partial u}{\partial x}+\frac{\partial C}{\partial y}\frac{\partial u}{\partial y}\right). \tag{5}$$

$$X=v\left(A\frac{\partial u}{\partial x}+B\frac{\partial u}{\partial y}+D u\right),\qquad Y=v\left(B\frac{\partial u}{\partial x}+C\frac{\partial u}{\partial y}+E u\right). \tag{6}$$

If L is self-adjoint then, due to (10.6), the expression Λ simplifies to

$$\Lambda=A\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}+B\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}+\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right)+C\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}+\frac{\partial D u v}{\partial x}+\frac{\partial E u v}{\partial y}-F u v,$$

which is again *symmetric in the u and v.* This can be further simplified by taking the terms

$$\frac{\partial D u v}{\partial x}\quad\text{and}\quad\frac{\partial E u v}{\partial y}$$

over to the X, Y on the right side of (1). We then obtain:

$$(7)\qquad \Lambda = A\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + B\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right) + C\frac{\partial u}{\partial y}\frac{\partial v}{\partial y} - F u v,$$

$$(8)\qquad X = v\left(A\frac{\partial u}{\partial x} + B\frac{\partial u}{\partial y}\right), \qquad Y = v\left(B\frac{\partial u}{\partial x} + C\frac{\partial u}{\partial y}\right).$$

If we now integrate equation (1) over a region S with the boundary curve C we obtain the most general version of the *second form of Green's theorem:*

$$(9)\qquad \int v L(u)\,d\sigma + \int \Lambda(u, v, \dots)\,d\sigma = \int \{X\cos(n, x) + Y\cos(n, y)\}\,ds.$$

c) In order to investigate whether the solution of the differential equation for a given boundary condition is *unique* we proceed as in the case of the equation $\Delta u = 0$:

Assuming that for given values of u on the boundary curve C different solutions u_1, u_2 exist, we set as in (9)

$$u = v = u_1 - u_2$$

Then, because of the linearity of L, the first term on the left side of (9) vanishes. Also, because $v = 0$ on the curve C, we see by (6) that $X = Y = 0$ on C, so that the right side of (9) vanishes. Thus (9) becomes

$$(10)\qquad \int \Lambda(u, u, \dots)\,d\sigma = 0.$$

If we restrict ourselves to the *self-adjoint* case and introduce the abbreviations $\xi = \partial u/\partial x$, $\eta = \partial u/\partial y$ we obtain

$$(11)\qquad \Lambda(u, u, \dots) = \begin{cases} \xi^2 + \eta^2 - F u^2 & \text{according to (4)} \\ A\xi^2 + 2B\xi\eta + C\eta^2 - F u^2 & \text{according to (7)} \end{cases}$$

The upper line of (11) contradicts (10) if $F(x,y)$ is *negative* throughout S; the second line contradicts (10) if the quadratic form $A\xi^2 + 2B\xi\eta + C\eta^2$ is *definite* and $F(x,y)$ has the *opposite* sign throughout S. In both these cases we conclude from (10) that

$$(12)\qquad u = 0 \quad \text{and hence} \quad u_1 = u_2$$

that is the *uniqueness of the boundary value problem.*

This is identical with the *non-existence of "eigenfunctions"* (see Chapter V). In particular, for the self-adjoint differential equation in the normal form $\Delta u + F u = 0$, where $F = \text{const.} = \pm k^2$, we see that the differential equation

$$(13)\qquad \Delta u - k^2 u = 0$$

has no *eigenfunctions* in contrast to the equation of the oscillating membrane

$$\Delta u + k^2 u = 0\,, \tag{13a}$$

where the eigenfunctions are of basic interest.

The fact that for the self-adjoint differential equation of elliptic type the *uniqueness question for the boundary value problem* or *the question about the existence of eigenfunctions* can be decided quite generally on the basis of the second form of Green's theorem, explains the preferred role of these differential equations in mathematical physics.

In order to stress the physical importance of equation (13) we remark that as the *Yukawa meson equation* it plays the same role in *nuclear physics* as is played by the potential equation in *electron physics*.

II.3. From the conditions a), b), c) of the exercise we obtain

$$\begin{cases} \text{for } 0 < x < \xi \ldots G_- = -\left(1 - \frac{\xi}{l}\right) x\,, \\ \text{for } \xi < x < l \ldots G_+ = -\left(1 - \frac{x}{l}\right) \xi\,. \end{cases} \tag{1}$$

On the other hand the solution of the "boundary value problem" for u is obviously

$$u = u_0 + (u_l - u_0)\,\frac{x}{l}\,. \tag{2}$$

Figure 35 represents the lines for G and u. Verify that Green's equation (10.12) for one dimension

$$u_\xi = u_l \left(\frac{\partial G_+}{\partial x}\right)_l - u_0 \left(\frac{\partial G_-}{\partial x}\right)_0 \tag{3}$$

II.4. According to equation (2) of the exercise, $v = v_0(t)$ is given for $x = 0$ and $t > 0$; hence the required Green's function G must satisfy the condition:

$$G = 0 \quad \text{for} \quad x = 0. \tag{1}$$

This function is obtained from the principal solution $V(x, t; \xi, \tau)$, equation (12.16), through reflection on the line $x = 0$ (see also §13):

$$\begin{aligned} G &= V(x, t;\ \xi, \tau) - V(x, t;\ -\xi, \tau) \\ &= \{4\pi k(\tau - t)\}^{-\frac{1}{2}} \left(\exp\left\{\frac{-(x-\xi)^2}{4k(\tau - t)}\right\} - \exp\left\{\frac{-(x+\xi)^2}{4k(\tau - t)}\right\}\right), \end{aligned}$$

and hence for $x = 0$:

$$\frac{\partial G}{\partial x} = \frac{\xi}{2\sqrt{\pi}}\,\{k(\tau - t)\}^{-\frac{3}{2}} \exp\left\{\frac{-\xi^2}{4k(\tau - t)}\right\}. \tag{2}$$

This value is to be substituted in (12.18) for $\partial V/\partial x$, so that we obtain the following simplification. On the right side we have to cancel the first term, since according to equation (4) of the exercise we have

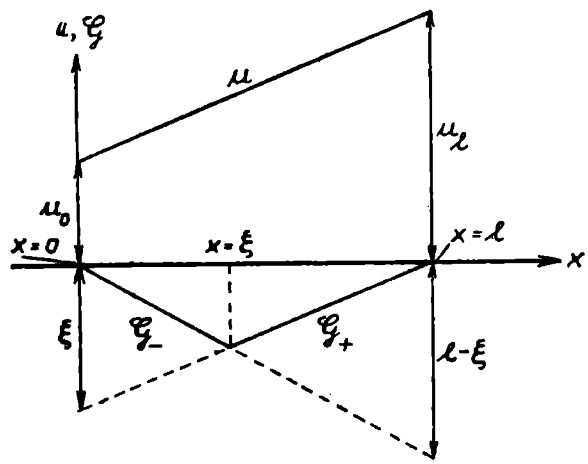


Fig. 35. Behavior of Green's function G with the source point $x = \xi$ and behavior of the potential function u with the boundary values u_0, u_l. The figure at the same time indicates, by the dotted extensions, an elementary construction of the ordinate of G at the point $x = \xi$ coincides with (2) if we use the values of G from (1).

$V = 0$ for $t = 0$. In the second term the part corresponding to $x_1 = \infty$ vanishes, so that only the term corresponding to $x_0 = 0$ remains, which according to (12.18) is to be taken negative; in this term the part multiplied by V vanishes because of (1). Hence, due to equation (2) of the exercise we have:

$$v(\xi,\tau) = \frac{\xi}{2\sqrt{\pi k}} \int_0^\tau v_0(t) \exp\left\{\frac{-\xi^2}{4k(\tau-t)}\right\} \frac{dt}{(\tau-t)^{\frac{3}{2}}}. \tag{3}$$

If instead of t we substitute the variable of integration $p = \xi/\sqrt{4k(\tau-t)}$ then we obtain

$$v(\xi,\tau) = \frac{2}{\sqrt{\pi}} \int_{\xi/\sqrt{4k\tau}}^{\infty} v_0\left(\tau - \frac{\xi^2}{4kp^2}\right) e^{-p^2}\,dp \tag{4}$$

as the final solution of the problem.

In order to discuss (4) we expand

$$v_0\left(\tau - \frac{\xi^2}{4kp^2}\right) = v_0(\tau) - \frac{v_0'(\tau)}{1!}\frac{\xi^2}{4kp^2} + \frac{v_0''(\tau)}{2!}\frac{\xi^4}{(4kp^2)^2} - \cdots. \tag{5}$$

Replacing the variables ξ, τ by x,t then we obtain from (4):

$$v(x,t) = v_0(t)\, I_0(z) - \frac{x^2}{4k}\frac{v_0'(t)}{1!} I_1(z) + \frac{x^4}{16k^2}\frac{v_0''(t)}{2!} I_2(z) - \cdots, \tag{6}$$

with the abbreviations

$$z = \frac{x}{\sqrt{4kt}}, \qquad I_0(z) = \frac{2}{\sqrt{\pi}}\int_z^\infty e^{-p^2}\,dp, \qquad I_n(z) = \frac{2}{\sqrt{\pi}}\int_z^\infty e^{-p^2}\frac{dp}{p^{2n}}. \tag{7}$$

Here $I_0(z)$ is essentially the well known and frequently tabulated "error integral." We have for $z \to \infty$

$$I_0(0) = 1, \quad I_0(z) \sim \frac{e^{-z^2}}{\sqrt{\pi}\, z}. \tag{8}$$

The corresponding statements for $I_n(z)$ are

$$I_n(z) \cdot z^{2n} \to 0, \qquad I_n(z) \cdot z^{2n} \sim \frac{e^{-z^2}}{\sqrt{\pi}\, z}. \tag{8a}$$

The expansion (6) is valid for $z < 1$. For $z \gg 1$ we have:

$$v(x,t) \sim \frac{e^{-z^2}}{z} \to 0. \tag{9a}$$

The transition between these limiting laws takes place at $z = 1$, that is, for

$$x \sim \sqrt{4kt}. \tag{9b}$$

The plate flow investigated here is a useful preparation for recent investigations of the turbulence problem (see v. II, §38).

III.1. According to (13.1) we write

$$u(x,t) = \int_0^\infty \{f(\xi)\, U(x,\xi) + f(-\xi)\, U(x,-\xi)\}\, d\xi. \tag{1}$$

Here $f(\xi)$ is given and we have to find $f(-\xi)$. For $x = 0$ we have

$$(4\pi k t)^{\frac{1}{2}}\, u(0,t) = \int_0^\infty \{f(\xi) + f(-\xi)\}\, e^{-\frac{\xi^2}{4kt}}\, d\xi, \tag{2}$$

$$(4\pi k t)^{\frac{1}{2}} \left(\frac{\partial u(x,t)}{\partial x}\right)_{x=0} = -\int_0^\infty \{f(\xi) - f(-\xi)\}\, \frac{d}{d\xi}\, e^{-\frac{\xi^2}{4kt}}\, d\xi. \tag{3}$$

If we assume that $f(\xi)$ can be made continuous at $\xi = 0$, that is, that

$$\lim_{\xi \to 0} f(-\xi) = \lim_{\xi \to 0} f(+\xi) = f_0, \qquad \xi > 0, \tag{4}$$

and if we transform the integral in (3) by integrating by parts, then the resulting term free of the integral sign vanishes and the right side of (3) becomes

$$\int_0^\infty \frac{d}{d\xi}\{f(\xi)-f(-\xi)\}\, e^{-\frac{\xi^2}{4kt}}\, d\xi. \tag{4a}$$

The condition imposed in the exercise for $x = 0$ is satisfied by the integrands in (2) and (4a). Considering the fact that here $\partial/\partial n$ is the same as $-\partial/\partial x$, we write:

$$\left(\frac{d}{d\xi}+h\right)f(-\xi) = X, \qquad X = \left(\frac{d}{d\xi}-h\right)f(\xi). \tag{5}$$

The differential equation for $f(-\xi)$ obtained in this way becomes integrable if we multiply by $\exp(h\,\xi)$ and yields

$$f(-\xi) = f(\xi) - 2\,h\,e^{-h\xi}\int_0^\xi e^{h\eta}\, f(\eta)\, d\eta. \tag{6}$$

This expression for $f(-\xi)$ is to be substituted in (1). Verify that we obtain the representation of G in (13.15) if in (1) we now specialize f to a δ-function.

III.2. We are dealing with Green's theorem (16.6) from which we are to deduce the normalizing integral (6.3a) by the limit process $\lambda_m \to \lambda_n$. Since this integral assumes the form 0/0 we apply de l'Hospital's rule by first differentiating the numerator and denominator with respect to λ_m and then setting $\lambda_m = \lambda_n$:

$$\frac{\left.\dfrac{du_n}{d\lambda_n}\dfrac{du_n}{dx} - u_n\dfrac{d}{d\lambda_n}\dfrac{du_n}{dx}\right|_{x=0}^{x=l}}{2\,k_n\dfrac{dk_n}{d\lambda_n}}.$$

The numerator must be computed for $x = l$ only, since for $x = 0$ it vanishes even before we pass to the limit.

According to (16.5) and (16.5a) we obtain

$$\int_0^l u_n^2\, dx = \frac{l}{2}\left(\cos^2\lambda_n\pi + \sin^2\lambda_n\pi - \frac{1}{\lambda_n\pi}\sin\lambda_n\pi\cos\lambda_n\pi\right),$$

which coincides with (6.4a) if we specialize our present l to π.

We performed this calculation mainly in order to be able to use it as a model in later cases where the normalizing integral cannot be integrated in an elementary manner.

III.3. In the stationary case equation (16.11) becomes

$$\frac{d^2u}{dx^2} = \lambda^2 u, \qquad \lambda = \sqrt{\frac{2h}{b}}. \tag{1}$$

From this we obtain as the general solution

$$u = A e^{\lambda x} + B e^{-\lambda x};$$

The coefficients A and B are computed from the values

$$u = u_1 \text{ for } x = 0,\ u = u_3 \text{ for } x = l.$$

We obtain

(2) $$u = \frac{u_3 \sinh \lambda x + u_1 \sinh \lambda(l - x)}{\sinh \lambda l}$$

and, setting $x = l/2$,

$$u_2 = \frac{(u_1 + u_3) \sinh \lambda l/2}{\sinh \lambda l} = \frac{u_1 + u_3}{2 \cosh \lambda l/2}.$$

The symbol q, which was introduced in the exercise now becomes:

(3) $$q = \cosh \lambda l/2.$$

From this we obtain a quadratic equation for $\exp(\lambda l/2)$, which yields

(4) $$\frac{\lambda l}{2} = \log\left(q + \sqrt{q^2 - 1}\right).$$

According to (1) we also have h expressed in terms of q. According to (13.5) h stands for the ratio of the exterior heat conductivity (which was denoted there by $4aT_0^3$) to the interior heat conductivity $\varkappa$. For $q = 1$ we obtain from (4) that $h = 0$ as required in the statement of the exercise.

III.4. In the stationary state the energy extracted from the. element of the rod (length dx, cross-section q) through heat conduction must equal the Joule heat generated in the same element. If we express the current i in terms of the potential difference V then we obtain as the differential equation of the stationary state:

$$\frac{d^2u}{dx^2} = -a, \quad a = \frac{V^2}{l^2}\frac{\sigma}{\varkappa}.$$

Due to the boundary conditions at the ends of the rod the integration of this differential equation yields a symmetric parabola as the graph of the temperature process. We determine a in terms of the maximal temperature U and obtain:

$$\frac{\varkappa}{\sigma} = \frac{1}{8}\frac{V^2}{U}.$$

Thus, through measurement of V and U we can verify the empirical law of Wiedemann and Franz which, according to the theory of metal electrons, asserts that

$$\frac{\varkappa}{\sigma}=\frac{\pi^2}{3}\left(\frac{k}{e}\right)^2 T,$$

where T = the absolute temperature, k = the Boltzmann constant, and e = the electron charge.

IV.1. a) For integral n we can expand the function $\exp(i\varrho\cos w)$ of (19.18) in the well known power series. As the coefficients of ϱ^k we then obtain

$$(1)\qquad a_k=\frac{e^{i(k-n)\pi/2}}{2\pi\, k!\, 2^k}\int_{-\pi}^{+\pi}(e^{iw}+e^{-iw})^k e^{inw}\,dw.$$

If we perform the binomial expansion under the integral sign then only one term remains upon integration, and even that term remains only for $k-n\geqq 0$. This result agrees with (19.34).

b) For non-integral n (ϱ is assumed real) we substitute in (19.14)

$$(2)\qquad t=\frac{\varrho}{2}e^{-i(w-3\pi/2)},\qquad dw=i\frac{dt}{t}.$$

The path W_0, which may be assumed rectangular, is then transformed into the loop of Fig. 37a which starts from $+\infty$, circles the origin in clockwise direction, and returns to $+\infty$ according to the scheme

$$w=i\infty-\frac{\pi}{2},\quad -\frac{\pi}{2},\quad 0,\quad +\frac{\pi}{2},\quad \pi,\quad \frac{3\pi}{2},\quad i\infty+\frac{3\pi}{2};$$

$$t=+\infty\quad,\quad e^{2\pi i}\frac{\varrho}{2},\quad e^{3\pi i/2}\frac{\varrho}{2},\quad e^{i\pi}\frac{\varrho}{2},\quad e^{i\pi/2}\frac{\varrho}{2},\quad \frac{\varrho}{2},\quad +\infty$$

Equation (19.14) then becomes

$$(3)\qquad I_n(\varrho)=-\frac{e^{i\pi n}}{2\pi i}\left(\frac{\varrho}{2}\right)^n\int_S e^{-t+\frac{\varrho^2}{4t}}t^{-n-1}\,dt.$$

If we now expand in $\exp\left(\frac{\varrho^2}{4t}\right)$ we again obtain the series (19.34), provided we use the following general definition of the Γ-function:

$$(4)\qquad \frac{1}{\Gamma(x+1)}=\frac{e^{i\pi(x+1)}}{2\pi i}\int_S e^{-t}t^{-x-1}\,dt;$$

We can easily verify that this definition coincides with the elementary definition $\Gamma(x+1)=x!$ for integral x by forming residues for $t=0$.

IV.2. In order to complete the investigation of the real part of $i\varrho\cos w$ (p. 86) we compute for complex $\varrho=|\varrho|e^{i\Theta}$ the quantity

(1) $$X = \mathbf{Re}\,(i\,\varrho \cos w) = \frac{|\varrho|}{2}\,[\sin(p-\Theta)\,e^{q} - \sin(p+\Theta)\,e^{-q}].$$

In order that X become negative in the infinite part of the upper half of the w-plane ($q \gg 1$) we must have

(2) $$\sin(p-\Theta) < 0.$$

The shaded strip

$$-\pi < p < 0$$

of Figs. 18 and 19 then shifts into the strip,

$$-\pi + \Theta < p < \Theta,$$

that is, the strip is shifted to the right or left in Fig. 18 according as θ increases or decreases. The opposite holds in the lower half of the w-plane, where according to (1) we must replace (2) by

(3) $$\sin(p+\Theta) > 0.$$

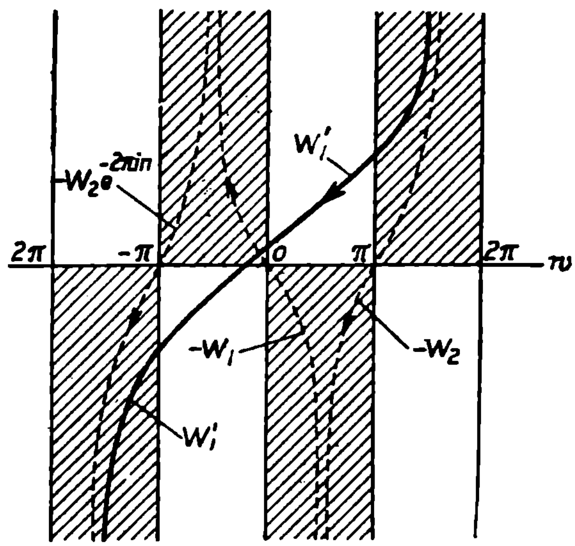


Fig. 36a. Distortion of the path of integration W_1 of H_n^1 into W_1' for a full circuit around the origin.

For $0 < \Theta < \pi$ (upper half of the positive ϱ-plane) the shaded regions of the upper and lower half planes have finite segments of the real axis in common, so that the path W_1 can be situated entirely within the shaded region. From this follows, without the use of the asymptotic formula (19.55), that H^1 vanishes for $\varrho \to \infty$ in the upper half plane. Due to this shift of the shaded regions we also see that for $0 > \Theta > -\pi$ (lower half of the ϱ-plane) the path W_1 must necessarily lead across the non-shaded region so that H^1 becomes infinite as $\varrho \to \infty$. The opposites of both these statements holds for W_2 and H^2.

Figure 36a illustrates the effect of this shift on a full circuit of ϱ around the origin with respect to the path W_1. The beginning of W_1 has been shifted by 2π, the end by -2π; thus W_1 has been distorted into W_1'. However W_1' can be decomposed into three partial paths of the same character as the original W_1, W_2 following the symbolic equation

$$W_1' = -W_2 - W_1 - W_2 e^{-2\pi i n}. \tag{4}$$

Here $-W_1$ is the middle one of the three shaded partial paths of the figure and it differs from W_1 in orientation only; $-W_2$ is the path on the right; the path on the left is obtained from W_2 by replacing w by $w-2\pi$, thereby changing the factor $\exp(inw)$ in the integrand by the factor $e^{-2\pi i n}$. Thus, as a result of (4) we obtain:

$$H_n^1(\varrho e^{2\pi i}) = -H_n^1(\varrho) - H_n^2(\varrho)(1+e^{-2\pi i n}). \tag{5}$$

When n is an integer this becomes

$$H_n^1(\varrho e^{2\pi i}) = -H_n^1(\varrho) - 2H_n^2(\varrho), \tag{6}$$

which we can rewrite in the form:

$$H_n^1(\varrho e^{2\pi i}) - H_n^1(\varrho) = -2\{H_n^1(\varrho) + H_n^2(\varrho)\} = -4I_n(\varrho). \tag{6a}$$

This change in H_n^1 of $4I_n$ together with the relation $H^1 = I + iN$ correspond to the change in $\log\varrho$ of $2\pi i$ in the formula (19.47) for N.

Figure 36b represents the correspondingly distorted path W_2' of H^2 for a full positive circuit of ϱ around the origin. This path can be decomposed into five partial paths of the same character as W_1, W_2 following the symbolic equation:

$$W_2' = W_2 + W_1 + W_1 e^{2\pi i n} + W_2 e^{-2\pi i n} + W_2 e^{+2\pi i n}. \tag{7}$$

Here W_2 is the middle one of the five partial paths, W_1 is the path adjacent on the left, $W_1 e^{2\pi i n}$ is the path adjacent on the right, etc. Instead of (5) we now obtain

$$H_n^2(\varrho e^{2\pi i}) = H_n^2(\varrho)(1 + e^{-2\pi i n} + e^{+2\pi i n}) + H_n^1(\varrho)(1 + e^{2\pi i n}). \tag{8}$$

When n is an integer this becomes

$$H_n^2(\varrho e^{2\pi i}) = 3H_n^2(\varrho) + 2H_n^1(\varrho), \tag{9}$$

$$H_n^2(\varrho e^{2\pi i}) - H_n^2(\varrho) = 2(H_n^1(\varrho) + H_n^2(\varrho)) = 4I_n(\varrho). \tag{9a}$$

The change (9a) together with the relation $H^2 = I - iN$ again correspond to the charge in $\log\varrho$ of $2\pi i$ in (19.47).

The equations (5) and (8) are the so-called *circuit relations* of the Hankel functions for the angle increment $\Delta\Theta = 2\pi$. These relations correspond to the "relationes inter contiguas" which were established by Gauss for the hypergeometric functions (see §24). Just as the equations (6a), (9a) for integral n are obtained from (19.47), the general relations (5) and (8) can be derived from the representations (19.31) and (19.30).

The circuit relations can be generalized from the full circuits $\Delta\Theta = 2\pi\nu$ (ν = integer) to half-circuits $\Delta\Theta = \pi\nu$. For a later applica-

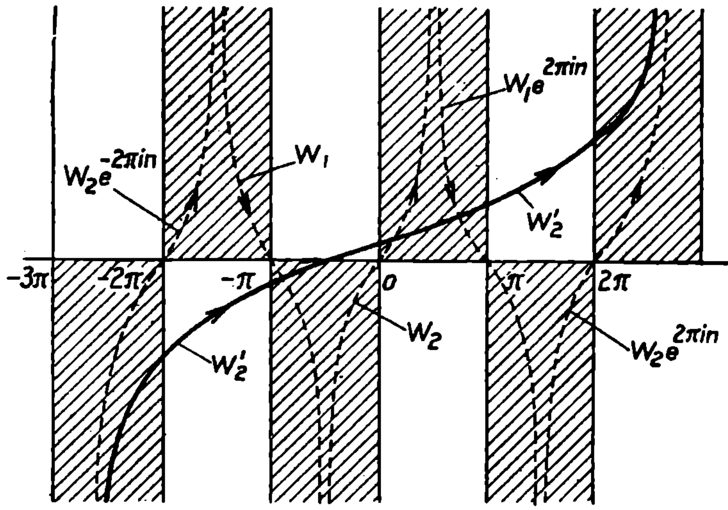


Fig. 36b. Distortion of the path W_2 of H_n^2 into W_2' for a full circuit around the origin.

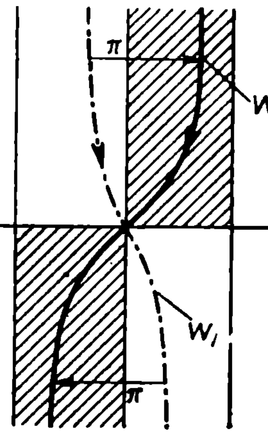


Fig. 36c. The half-circuit relation for H_0^1.

tion in §32 we discuss the half-circuit relation for $\nu = 1$ and $n = 0$. The relation reads

$$H_0^1(\varrho\, e^{i\pi}) = -H_0^2(\varrho). \tag{10}$$

For a proof, a look at Fig. 36c suffices. For real ϱ the path W_1 (indicated by dots and dashes in the figure) leads from the region $-\pi < p < 0$ to the region $0 < p < +\pi$; for the present argument $\varrho e^{i\pi}$ this path has been shifted by $+\pi$ in the upper part and by $-\pi$ in the lower part, as indicated by the arrows in the figure. The path W_1' obtained in this manner is identical with the path W_2 for H^2. But this is the statement of equation (10).

A relation which is analogous to (10) is obtained if we replace ϱ by $\varrho e^{-i\pi}$:

$$H_0^2(\varrho\, e^{-i\pi}) = -H_0^1(\varrho). \tag{10a}$$

Considering the factor exp (inw) of the integrand we can generalize (10) to

$$H_n^1(\varrho\, e^{i\pi}) = -e^{-n\pi i} H_n^2(\varrho), \tag{11}$$

for arbitrary n, or (replacing ϱ by $\varrho^{-i\pi}$)

$$H_n^2(\varrho\, e^{-i\pi}) = -e^{+n\pi i} H_n^1(\varrho). \tag{11a}$$

These half-circuit relations can also be derived directly from the equations (19.30) and (19.31) with the help of the equations

(12) $$I_n(\varrho \cdot e^{i\pi\nu}) = e^{i\pi n\nu} I_n(\varrho), \quad I_{-n}(\varrho \cdot e^{i\pi\nu}) = e^{-i\pi\nu n} I_{-n}(\varrho),$$

which follow from (19.34).

These relations become very simple if we write them for $\psi_n(\varrho)$ and $\zeta_n(\varrho)$ in which the corresponding Hankel and Bessel functions of index $n+\frac{1}{2}$ with integral n are multiplied by $\sqrt{2\varrho/\pi}$; namely we have

(13) $$\zeta_n^{1,2}(\varrho\, e^{\pm i\pi}) = (-1)^{n+1}\zeta_n^{2,1}(\varrho), \qquad \psi_n(\varrho\, e^{\pm i\pi}) = (-1)^{n+1}\psi_n(\varrho).$$

In the representation (19.22) for $H_0^1(\varrho)$ we substitute

$$x = i\varrho\cos w, \qquad dw = \frac{i\,dx}{\sqrt{x^2+\varrho^2}}.$$

The path W_1, which for convenience is to be taken rectangular, is then transformed into the x-plane according to the scheme

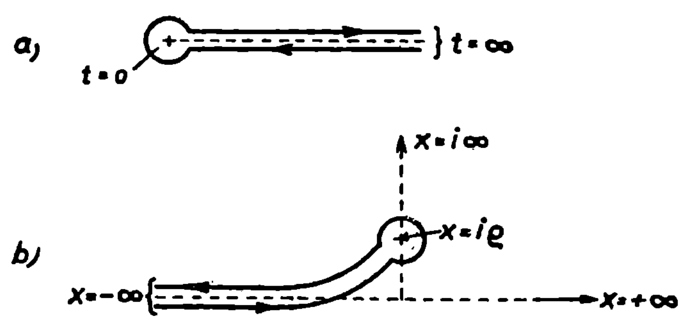


Fig. 37a. The loop integrals for $1/\Gamma(x+1)$. b. The loop integral for $H_0^1(\varrho)$ for small ϱ.

$$w = -\frac{\pi}{2}+i\infty, \quad -\frac{\pi}{2}, \quad 0, \quad +\frac{\pi}{2}, \quad \frac{\pi}{2}+i\infty,$$

$$x = -\infty \quad , \quad 0, \quad i\varrho, \quad 0, \quad -\infty \quad .$$

We thus have the loop integral of Fig. 37b that begins at the negative infinite end of the real x-axis, circles the point $x = i\varrho$ and returns to the negative infinite end of the real axis; the orientation of this loop is controlled by a small displacement of the real branch of W_1. From (19.22) we then obtain:

(1) $$H_0^1(\varrho) = \frac{2i}{\pi}\int_{-\infty}^{+i\varrho} \frac{e^x\,dx}{\sqrt{x^2+\varrho^2}}.$$

This integral is obtained by the combination of the two branches of Fig. 37b, where the originally negative sign of the returning branch has been reversed through the complete circuit around the branch point

$x = i\varrho$. Through integration by parts with respect to x we obtain

$$(2)\qquad H_0^1(\varrho) = \frac{2i}{\pi}\left\{e^x \log\left(x + \sqrt{x^2+\varrho^2}\right)\right\}_{x=i\varrho} - \frac{2i}{\pi}\int_{-\infty}^{i\varrho} e^x \log\left(x+\sqrt{x^2+\varrho^2}\right) dx\,.$$

Substituting $x = i\varrho$ in the first term and letting $\varrho \to 0$ in both terms we obtain

$$(3)\qquad \lim_{\varrho\to 0} H_0^1(\varrho) = \frac{2i}{\pi}\log i\varrho - \frac{2i}{\pi}\int_{-\infty}^{0} e^x \log 2x\, dx\,.$$

With the substitution $x = -t$ the last integral becomes

$$(4)\qquad \log(-2) + \int_0^\infty e^{-t}\log t\, dt = \log(-2) - C = \log\left(\frac{-2}{\gamma}\right).$$

Here C and γ are the quantities defined in (19.41a) (check using the Laplace integral for the Γ-function). Combining (4) and (3) we obtain from (1):

$$(5)\qquad \lim_{\varrho\to 0} H_0^1(\varrho) = \frac{2i}{\pi}\left(\log\frac{\gamma\varrho}{2} - \frac{i\pi}{2}\right) = \frac{2i}{\pi}\log\frac{\gamma\varrho}{2} + 1\,.$$

Due to the relation

$$H_0 = I_0 + i N_0,\quad \lim_{\varrho\to 0} H_0 = 1 + i \lim_{\varrho\to 0} N_0$$

equation (5) coincides with the equation (19.48) for N.

IV.4. 1. If we neglect $1/\varrho$ in the differential equation (19.11) we obtain $H_n^1 = A\, e^{i\varrho}$ (A = constant of integration; the solution involving $e^{-i\varrho}$ corresponds to H_n^2).

2. We now consider A not as a constant but as a "slowly varying function of ϱ" such that A'', A'/ϱ and A/ϱ^2 can be neglected. This yields a differential equation for $A(\varrho)$, from which we obtain $A = B/\sqrt{\varrho}$. The normalizing constant B cannot, of course, be determined in this manner.

IV.5. a) According to the equations (1.12), (22.14), (22.31) we obtain as the coefficients C_m and A_{mn} of the exercise:

$$C_m = \frac{1}{2\pi}\int_{-\pi}^{+\pi} f(\vartheta_0, \varphi_0)\, e^{-im\varphi_0}\, d\varphi_0,$$

$$A_{mn} = \frac{1}{N_n^m}\int_{-\pi}^{+\pi} C_m(\vartheta_0)\, P_n^m(\cos\vartheta_0)\sin\vartheta_0\, d\vartheta_0$$

$$= \frac{n+\frac{1}{2}}{2\pi}\frac{(n-m)!}{(n+m)!}\int_0^{\pi}\sin\vartheta_0\,d\vartheta_0\int_{-\pi}^{+\pi}d\varphi_0 f(\vartheta_0,\varphi_0)\,P_n^m(\cos\vartheta_0)\,e^{-im\varphi_0}.$$

b) From the scheme for $f(\vartheta,\varphi)$ in the exercise we see that if we multiply $f(\vartheta_0,\varphi_0)$ by

$$Y_{\nu\mu} = P_\nu^\mu(\cos\vartheta_0)\,e^{-i\mu\varphi_0}$$

and integrate with respect to φ_0 then we obtain

$$\int_{-\pi}^{+\pi} f(\vartheta_0,\varphi_0)\,Y_{\nu\mu}\,d\varphi_0 = 2\pi\sum_n A_{n\mu}\,P_n^\mu(\cos\vartheta_0)\,P_\nu^\mu(\cos\vartheta_0),$$

Integrating with respect to $\sin\vartheta_0\,d\vartheta_0$ from (22.14) and (22.31) we obtain the result

$$\int\int f(\vartheta_0,\varphi_0)\,Y_{\nu\mu}\,d\sigma_0 = 2\pi\,N_\nu^\mu\,A_{\nu\mu},$$

After a change in notation (ν, μ instead of n,m) this coincides with the expression for A_{mn} in a) except for the order of summation (see Fig. 38).

In a) the horizontal strips $|m| \leq n < \infty$ are summed in the vertical direction, and in b) the vertical strips $-n \leq m \leq +n$ are summed in the horizontal direction. In both cases the total domain of summation is bounded by the lines $n = \pm m$; thus the sums are the same.

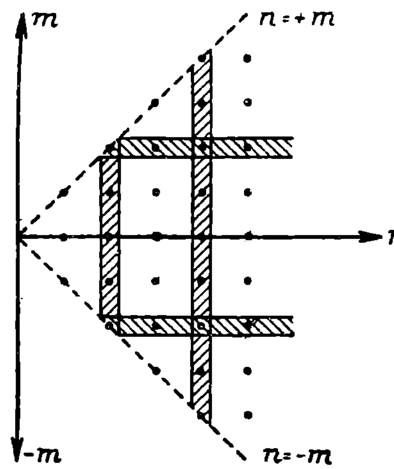


Fig. 38. The double sum in the number lattice of n,m; arranged in horizontal strips for case a) and in vertical strips for case b).

IV.6. Again we draw the 60° wedge of Fig. 17 together with its five reflected images, so that the original wedge lies symmetric with respect to the horizontal plane. For reasons of convenience we situate the center of inversion C on the horizontal line through the vertex O of the wedge and we let the circle of inversion (dotted circle in the figure) pass through O. Since the point at infinity is mapped into C and the points of intersection O and S_1 of the line 1 with the circle of inversion remain fixed, the position of the circle into which the straight line 1, $-$ 1 is transformed is determined by the points O,C,S_1. The arcs of the circle which correspond to the half lines 1 and $-$ 1 are again denoted by 1 and $-$ 1. The same holds for the line 2, $-$ 2 which is mapped into a circle of the same radius passing through the points O,C,S_2. The line 3, $-$ 3 goes into a circle of diameter OC in which the upper and lower semicircles correspond to the half lines $-$ 3 and 3.

Now the wedge 1,2 is mapped into the exterior of the circular diangle $C,S_1,1,O,2,S_2,C$; both regions are indicated by a shading of the boundary. We now seek the images of the reflected wedges. All these images are interiors of certain circular diangles (crescents); e.g., the wedge 2,3 is mapped into the crescent $C,S_2 2,0,3,C$, and the wedge -2, -1 is mapped into the small lens-like region $C,-2,O,-1,C$ in the center of our figure.

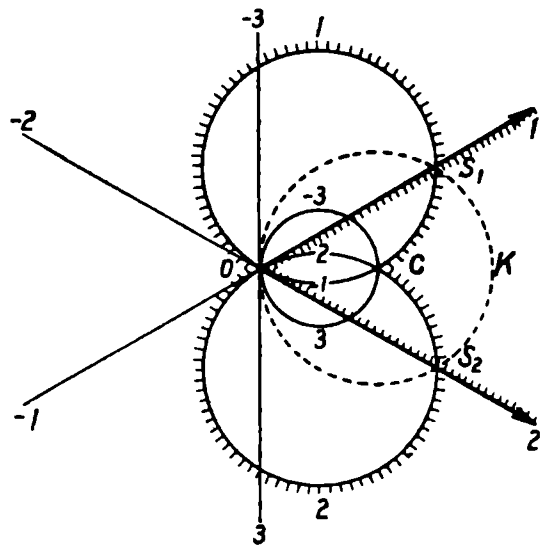


Fig. 39. The wedge 1,2 of the 60° angle is mapped by inversion into the exterior of $C,1,O,2,C$ of the intersecting spheres $1,-1$ and $2,-2$; the reflected wedges 2,3; $3,-1$; . . . are mapped into spherical crescents.

Up to now we have described the drawing as a *plane* figure and spoken of straight lines, circles, circular diangles, etc. However there is nothing that prevents us from interpreting the figure as *three-dimensional* and to speak of planes and spheres instead of straight lines and circles. These spheres are then situated with their centers in the plane of the drawing. The original wedge 1,2 then is mapped into the exterior of the two intersecting spheres which belong to the circles 1, -1 and 2, -2; in the same manner the reflected wedges correspond to the regions bounded by two of the spheres 1,2,3.

Just as before, we obtained *Green's function* for the wedge from the elementary reflections in Fig. 17, now in the case of the *potential equation* (but only in this case) we obtain Green's function for the corresponding circular or spherical regions by finding the "electric image" of the given pole upon inversion on the boundary circles or spheres 1,2,3 and by giving alternating signs to these poles. For the symmetric structure of our problem it suffices to have five such electric images in order to satisfy the boundary condition $u = 0$ on the boundary of each of the regions under consideration.

IV.7. a) In the inversion in the sphere K (broken line) of Fig. 40 all the infinite points of the reflecting planes ± 1, ± 2, ± 3, . . . are mapped into the center of inversion C; thus the planes $\pm$ 1 go into the spheres $+$ 1 and $-$ 1 which are tangent at C and have diameter equal to the radius a of the sphere of inversion. Here the exterior of the spheres $\pm$ 1 corresponds to the interior of the plate and the interior of these spheres corresponds to the exterior of the plate. The planes ± 2 are mapped into spheres which are again tangent at C but have a diameter of only $a/3$. The images of the planes ± 3 are in turn spheres which lie in the interior of the spheres ± 2, have the diameter $a/5$ and are tangent at C. The region bounded by two consecutive spheres in this sequence corresponds to a reflected image of the original plate.

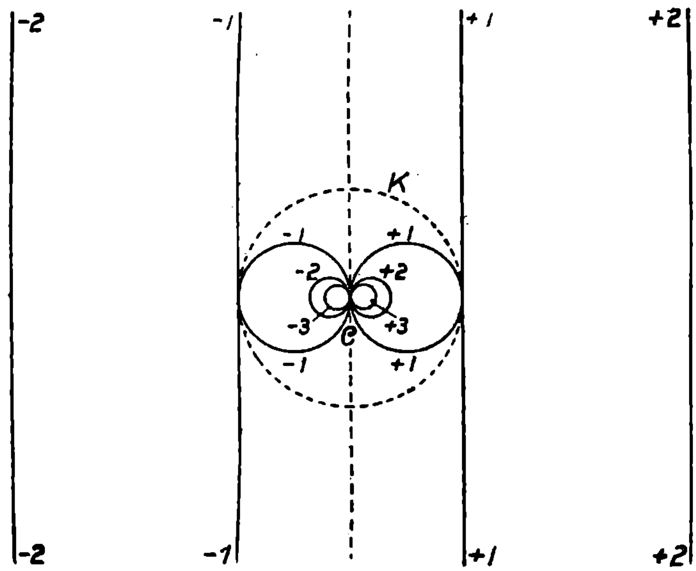


Fig. 40. Inversion of a plane parallel plate and its successive reflected images into a system of spheres tangent at the center of inversion C.

Green's function of potential theory for the exterior of two tangent spheres (e.g., the spheres ± 1 of our figure) can be deduced by inversion from Green's function for the plane parallel plate. The infinitely many image points of the arbitrarily prescribed pole of Green's function that arise in the inversion are situated in the successive spherical regions mentioned above, and they accumulate at the point C.

b) If the radii of the concentric spheres I and II are a and $2a$ then we may choose the radius of the sphere of inversion equal to a and place its center C on sphere I. Then sphere I is mapped into the plane E_{I}, and II is mapped into a sphere K_{II} of radius $2a/3$; the minimum distance between K_{II} and E_{I} is $a/6$. Conversely, E_{I} and K_{II} are mapped into the concentric spheres I and II.

For an arbitrary position of the non-intersecting plane E and sphere K we can proceed in the following way (kindly communicated to me by Caratheodory): from the center of K we drop the perpendicular L to E; from the foot F of L we draw tangents (of length t) to K and draw the auxiliary sphere H with center F and radius t. As the center of inversion we choose one of the points of intersection S of L and H. Then L is transformed into a straight line, H becomes a plane which is perpendicular to L, and E and K become spheres which are perpendicular to H and L and hence have their center at the point of intersection of H and L, that is, in the center of inversion. The radius of inversion remains arbitrary and determines only the size of the concentric spheres.

Instead of E and K we may also consider two arbitrary non-intersecting spheres K_1 and K_2 (see the last statement in the exercise). In order to transform them into two concentric spheres we start from the pencil of spheres $K_1 + \lambda K_2 = 0$. The pencil contains two spheres of radius 0, namely the two poles of the bipolar coordinate system. If we choose one of these poles as the center of inversion then all the spheres of the pencil, including K_1 and K_2 are mapped into *concentric* spheres.

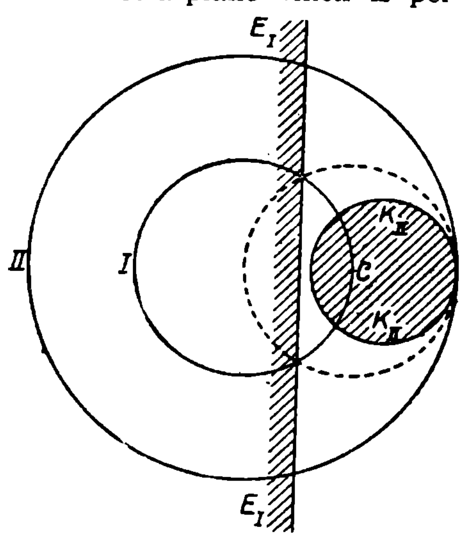


Fig. 41. Two concentric spheres I and II are transformed through inversion into the plane E_I and the sphere K_{II} (which are shaded in the figure).

IV.8. Let u_1 and u_2 be linearly independent solutions of the differential equation

$$L(u) = u'' + pu' + qu = 0$$

of second order, where p and q are arbitrary given functions of the independent variable ϱ. Then for $X = u_1u_2' - u_2u_1'$ we have:

$$u_1 L(u_2) - u_2 L(u_1) = \frac{dX}{d\varrho} + pX = 0,$$

hence $$X = C\,e^{-\int p\,d\varrho}, \quad C = \text{constant of integration.}$$

a) For the Bessel differential equation (19.11) we have and hence

$$X = C\,e^{-\log\varrho} = \frac{C}{\varrho}.$$

If we take $u_1 = H_n^1$, $u_2 = H_n^2$ then C is determined most simply from the asymptotic values (19.55), (19.56). We obtain

$$X = \frac{-4i}{\varrho\pi}, \qquad C = \frac{-4i}{\pi}.$$

Due to $I_n = \frac{1}{2}(H_n^1 + H_n^2)$, we see that the expression (I) in the exercise, where H is to stand for H^1, equals half the above X, so that

$$(\mathrm{I}) = -\frac{2i}{\varrho \pi} \quad ;$$

the sign is reversed if H is to stand for H^2.

The determination of C becomes somewhat less simple if we start from $\varrho = 0$ instead of $\varrho = \infty$.

b) In the differential equation (21.11a) we have $p = 2/\varrho$, and hence $X = C/\varrho^2$. If we take $u_1 = \zeta_n^1$, $u_2 = \zeta_n^2$, then according to (21.14) we have, for $\varrho \to \infty$,

$$\zeta_n^{1,2} = \frac{1}{\varrho} e^{\pm i[\varrho - (n+1)\pi/2]}, \qquad X = -\frac{2i}{\varrho^2}, \qquad C = -2i.$$

For the expression (II) we then obtain:

$$(\mathrm{II}) = \mp \frac{i}{\varrho^2} \quad ,$$

where the sign depends on whether we set ζ equal to ζ^1 or ζ^2.

V.1. a) Due to $I_n'(\lambda a) = 0$ the limit process of (20.9) yields the normalizing integral

$$N = -a \lim_{\varepsilon \to 0} \frac{I_n(\lambda a)\, I_n'(\lambda a + \varepsilon a)}{2\varepsilon} = -\frac{a^2}{2} I_n(\lambda a)\, I_n''(\lambda a)$$

instead of (20.19). With the help of the Bessel differential equation we obtain

$$N = \frac{1}{2\lambda^2}(\lambda^2 a^2 - n^2)\, I_n^2(\lambda a).$$

Hence if we "normalize $I_n(\lambda r)$ to 1" we obtain

$$\sqrt{\frac{2\lambda^2}{\lambda^2 a^2 - n^2}}\, \frac{I_n(\lambda r)}{I_n(\lambda a)}.$$

b) From the relation (21.11) between $\psi_n(\varrho)$ and $I_{n+\frac{1}{2}}(\varrho)$ we obtain for the present normalizing integral

$$N = \int_0^a \psi_n^2(kr)\, r^2\, dr = \frac{\pi}{2k} \int_0^a I_{n+\frac{1}{2}}^2(kr)\, r\, dr.$$

For the boundary condition

$$\psi_n'(ka) = \sqrt{\frac{\pi}{2ka}}\left(I_{n+\frac{1}{2}}'(ka) - \frac{1}{2ka} I_{n+\frac{1}{2}}(ka)\right) = 0$$

of the exercise we obtain by a limit process analogous to that of a)

$$N = -\frac{\pi}{2k}\frac{a^2}{2} I_{n+\frac{1}{2}}(ka)\left\{I''_{n+\frac{1}{2}}(ka) + \frac{1}{2ka} I'_{n+\frac{1}{2}}(ka)\right\}$$

with the help of the Bessel differential equation we may therefore write

$$N = \frac{a}{2k^2}\psi_n^2(ka)\{k^2a^2 - n(n+1)\};$$

thus the normalized form of ψ_n is

$$\Psi_n = \sqrt{\frac{2k^2/a}{k^2a^2 - n(n+1)}}\,\frac{\psi_n(kr)}{\psi_n(ka)}.$$

V.2. The proof follows from Green's theorem

$$(1) \qquad \int (u\Delta v - v\Delta u)\,d\tau = \int \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right)d\sigma,$$

if we set $v = 1/r$ where r stands for the distance of the point of integration from P. Due to the singularity of v at P we surround P in the customary manner by a sphere K_ϱ of radius $\varrho \to 0$. If we extend the integration on the left side over the region bounded by K_a and K_ϱ, then the left side vanishes and the right side becomes the sum of the surface integrals over K_a and K_ϱ (in both cases n stands for the exterior normal to the region). By letting $\varrho \to 0$ in the integral over K_ϱ we obtain from (1)

$$(2) \qquad 0 = 4\pi u_P - \frac{1}{a^2}\int_{K_a} u\,d\sigma - \frac{1}{a}\int_{K_a}\frac{\partial u}{\partial n}\,d\sigma.$$

The third term on the right here vanishes since throughout the interior of S we have $\Delta u = 0$. Thus equation (2) proves the theorem of the arithmetic mean.

V.3. From equation (27.14) and the condition $u = U$ on the sphere $r_0 = a$ we obtain

$$(1) \qquad 2\pi U = \sum_n \sum_m A_{nm}\Pi_n^m(\cos\vartheta_0)\,e^{-im\varphi_0};$$

Multiplying by $e^{i\mu\varphi_0}$ and integrating with respect to φ_0 from 0 to π we obtain:

$$(2) \qquad \int_0^{2\pi} U e^{i\mu\varphi_0}\,d\varphi_0 = \sum_n A_{n\mu}\Pi_n^\mu(\cos\vartheta_0);$$

multiplying by $\Pi_\nu^\mu(\cos\vartheta_0)\sin\vartheta_0$ and integrating with respect to ϑ_0 from

0 to π we have:

(3) $$\int_0^{\pi}\int_0^{2\pi} U\, \Pi_\nu^\mu(\cos\vartheta_0)\, e^{i\mu\varphi_0} \sin\vartheta_0\, d\vartheta_0\, d\varphi_0 = A_{\nu\mu}\,,$$

which coincides with (27.13a) except for notation.

Comparing the r-dependence of (27.13) and (27.14) we obtain

(4) $$-a^2 \sum_l \frac{1}{k_{nl}} \Psi_n(k_{nl} r_0)\, \Psi_n'(k_{nl} a) = \left(\frac{r_0}{a}\right)^n,$$

The summation extends over all the roots $k = k_n$ of the equation $\Psi_n(ka) = 0$, which are the same as the roots of the equation $\psi_n(ka) = 0$. In order to determine the Ψ_n in terms of the ψ_n we use equation (21.11)

(5) $$\psi_n(x) = \sqrt{\frac{\pi}{2x}}\, I_{n+\frac{1}{2}}(x)$$

and the relation (20.19) (which holds for non-integral n, too)

(5a) $$\int_0^a [I_{n+\frac{1}{2}}(kr)]^2\, r\, dr = \frac{a^2}{2}[I'_{n+\frac{1}{2}}(ka)]^2,$$

where k is a root of $I_{n+\frac{1}{2}}(ka) = 0$, and hence a root of $\psi_n(ka) = 0$. From (5) and (5a) we obtain:

(6) $$\int_0^a [\psi_n(kr)]^2\, r^2\, dr = \frac{a^3}{2}[\psi_n'(ka)]^2.$$

If we now set $\Psi_n = N\psi_n$ and impose the condition

(7) $$\int_0^a \Psi_n^2(kr)\, r^2\, dr = 1,$$

we obtain

$$N^2 = 2/a^3\, [\psi_n'(ka)]^2.$$

Rewriting the equation (4) in terms of ψ_n, and adopting the notation $\alpha = r_0/a$, we obtain:

(8) $$2\sum_l \frac{\psi_n(k_{nl}\,\alpha a)}{a\, k_{nl}\, \psi_n'(k_{nl} a)} = -\alpha^n.$$

For $n = 0$ we have according to (21.11)

$$\psi_0(x) = \frac{\sin x}{x};$$

therefore

$$\psi_0(ka) = 0 \quad \text{for} \quad k = k_{0l} = \frac{l\pi}{a}, \qquad l = \pm 1, \pm 2, \ldots$$

In this particularly simple case equation (8) becomes

(9) $$2\sum_{l=1}^{+\infty}(-1)^l\frac{\sin l\pi\alpha}{l} = -\pi\alpha.$$

For $\alpha = \frac{1}{2}$ this yields

$$1-\frac{1}{3}+\frac{1}{5}-\cdots=\frac{\pi}{4},$$

which is the Leibniz series (2.8). In general we obtain from (2.9) for $x = 2\pi\alpha$ a representation of the "saw-tooth profile"

$$\left.\begin{matrix}+\frac{1}{2}(\pi-x)\\ \\ -\frac{1}{2}(\pi+x)\end{matrix}\right\} = \sin x + \frac{\sin 2x}{2}+\frac{\sin 3x}{3}+\cdots \text{ for } \left\{\begin{matrix}0<x<+\pi,\\ \\ -\pi<x<0.\end{matrix}\right.$$

The reader is asked to verify this as a further exercise for chapter I. It is apparent, however, that equation (8) for $n > 0$ contains far deeper and more general analytic relations.

VI.1. a) Due to the fact that Π has the z-direction and depends only on z and $r^2 = x^2 + y^2$, we obtain from (31.4)

$$\mathbf{E}_x = \frac{\partial}{\partial x}\frac{\partial \Pi}{\partial z} = \frac{x}{r}\frac{\partial^2 \Pi}{\partial r\,\partial z},$$

$$\mathbf{E}_y = \frac{\partial}{\partial y}\frac{\partial \Pi}{\partial z} = \frac{y}{r}\frac{\partial^2 \Pi}{\partial r\,\partial z}.$$

For the form a) of the exercise we have

$$\frac{\partial \Pi}{\partial z} = \frac{z-h}{R}\frac{d}{dR}\frac{e^{ikR}}{R}+\frac{z+h}{R'}\frac{d}{dR'}\frac{e^{ikR'}}{R'}.$$

which vanishes when $z = 0$ since then $R = R'$. Hence the expressions for $\mathbf{E}_x$ and $\mathbf{E}_y$, which are obtained by differentiation with respect to x,y or r, also vanish for $z = 0$.

b) According to (31.4) we now have

$$\mathbf{E}_x = k^2\Pi + \frac{\partial}{\partial x}\frac{\partial \Pi}{\partial x},$$

$$\mathbf{E}_y = \frac{\partial}{\partial y}\frac{\partial \Pi}{\partial x}.$$

But since in the form b) Π vanishes for $z = 0$, we also have $\mathbf{E}_x$ and $\mathbf{E}_y$ vanishing for all points (x,y) on the earth's surface.

c) From (35.1) we obtain

$$\mathbf{E}_x = i\mu_0\omega\frac{\partial \Pi_z}{\partial y}, \quad \mathbf{E}_y = -i\mu_0\omega\frac{\partial \Pi_z}{\partial x}.$$

These derivatives vanish for $z = 0$, since in the form c) Π_z itself vanishes for $z = 0$.

d) From (35.1) we now have

$$\mathbf{E}_x \equiv 0, \qquad \mathbf{E}_y = i\,\mu_0\,\omega\,\frac{\partial \Pi_z}{\partial x}.$$

But according to the form d) we have

$$\frac{\partial \Pi_z}{\partial z} = \frac{z-h}{R}\,\frac{d}{dR}\,\frac{e^{ikR}}{R} + \frac{z+h}{R'}\,\frac{d}{dR'}\,\frac{e^{ikR'}}{R'}.$$

which vanishes for $z = 0$ since then $R' = R$.

VI.2. This exercise is instructive not only for the understanding of Zenneck waves, but also for the general knowledge of electromagnetic rotational fields and for their representation using complex operators.

From (32.20) according to the prescription (31.4), we obtain for the *air*

$$\text{(1)} \qquad \begin{aligned} \mathbf{E}_x &= -\,i\,p\sqrt{p^2-k^2}\,A\,k_E^2\,e^{ipx-\sqrt{p^2-k^2}\,z},\\ \mathbf{E}_z &= p^2\,A\,k_E^2\,e^{ipx-\sqrt{p^2-k^2}\,z}. \end{aligned}$$

These expressions, multiplied by the exponential time factor, represent an elliptic oscillation as known from optics. Due to the complex nature of the right sides of (1) the principal axes of the oscillation ellipse are oblique to the x- and z-direction. If we form the absolute values of $\mathbf{E}_x$ and $\mathbf{E}_z$ together with their negatives, then we obtain the limits between which $\mathbf{E}_x$ and $\mathbf{E}_z$ oscillate, in other words we obtain a rectangle circumscribed about the ellipse. The ratio of the sides of this rectangle is given by the absolute value of

$$\text{(2)} \qquad \frac{\mathbf{E}_x}{\mathbf{E}_z} = \frac{\sqrt{k^2-p^2}}{p} = \frac{1}{n}.$$

The value $1/n$ is obtained from the definitions (32.16a) and (32.2) of p and n. Because $|n| > |$ the rectangle is tall and narrow (See Fig. 42a).

On the other hand, due to (32.20) and (31.7), in the *earth* we have

$$\text{(3)} \qquad \begin{aligned} \mathbf{E}_x &= +\,i\,p\sqrt{p^2-p_E^2}\;\;A\,k^2\,e^{ipx+\sqrt{p^2-k_E^2}\,z},\\ \mathbf{E}_z &= \qquad\qquad\;\; p^2\,A\,k^2\,e^{ipx+\sqrt{p^2-k_E^2}\,z}. \end{aligned}$$

Hence again we have an elliptic oscillation that, is this time, situated in a rectangle with the ratio of sides given by the absolute value of

$$\text{(4)} \qquad \frac{\mathbf{E}_x}{\mathbf{E}_z} = \frac{\sqrt{k_E^2-p^2}}{p} = n,$$

where n is again obtained from (32.16a) and (32.2). Because $|n| > 1$ the rectangle is now broad and squat (See Fig. 42b). The present ellipse is traversed in the opposite sense of the former, as is seen from the reciprocity of the values n and $1/n$ in (4) and (2).

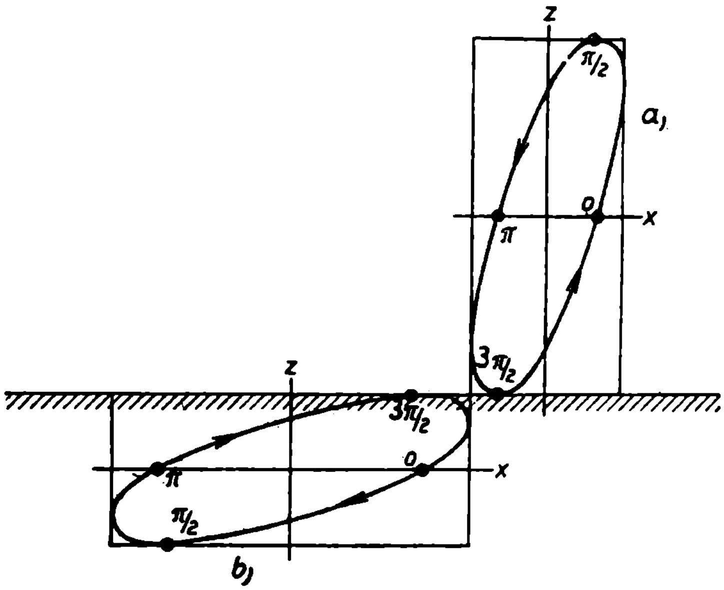


Fig. 42. The rotational field of the Zenneck wave a) in the air, narrow ellipse, b) in the earth, wide ellipse, congruent and of opposite orientation to each other.

If we think of the field in the air as pushing forward with its phase velocity in the positive x-direction, then the field in the earth appears to lag behind against the resistance there.

VI.3. The electric field strength at the antenna and in the direction of the antenna is $\mathbf{E} = \mathrm{Re}\{E e^{-i\omega t}\}$. At an antenna which is short compared to the wavelength this field strength does the work $\mathbf{E} j_t dt$ in the time dt. Hence according to equation (36.20) we have as the time average of work

$$W = j\, l \int_0^\tau \mathrm{Re}\{E\, e^{-i\omega t}\}\, \mathrm{Re}\{i\, e^{-i\omega t}\} \frac{dt}{\tau}.$$

where τ stands for the time of oscillation. With the method given on p. 271 we obtain

$$W = \frac{j\, l}{4} \int_0^\tau (E\, e^{-i\omega t} + E^* e^{+i\omega t})\, (i\, e^{-i\omega t} - i\, e^{+i\omega t}) \frac{dt}{\tau}$$

$$= \frac{j\, l}{4} (-\, i\, E + i\, E^*).$$

and hence also

(1) $$W = \frac{j\,l}{2}\mathrm{Re}\{-\,i\,E\}.$$

a) *Vertical antenna.* From (31.4) and the differential equation of Π we obtain for $\Pi = \Pi_z$ and $\mathbf{E} = \mathbf{E}_z$

$$E = k^2\,\Pi + \frac{\partial^2 \Pi}{\partial z^2} = -\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial \Pi}{\partial r}.$$

We express Π with the help of equation (32.9) and both e^{ikR}/R and $e^{ikR'}/R'$ are expressed with the help of (31.14). Since the r-dependence of these three terms is given by $I_0(\lambda\,r)$, the application of the operator

$$-\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}$$

under the integral sign yields the common factor $+\,\lambda^2 I_0(\lambda\,r)$, and hence at the point $r = 0$ of the antenna it yields the factor λ^2. Thus we obtain from (32.9):

(3) $$\begin{aligned}\mathrm{Re}\{-\,i\,E\} = \mathrm{Re}\Big\{-\,i\int_0^\infty (e^{-\mu|z-h|} + e^{-\mu(z+h)})\,\frac{\lambda^3\,d\lambda}{\mu}\\ +\,2\,i\int_0^\infty e^{-\mu(z+h)}\,\frac{\mu_E}{n^2\,\mu + \mu_E}\,\frac{\lambda^3\,d\lambda}{\mu}\Big\}.\end{aligned}$$

Since μ is real for $\lambda > k$ (see p. 273), the integral over $k < \lambda < \infty$ in the first line does not contribute to the real part and we can pass to $z = h$, that is, to the point of the antenna, without encountering difficulties of convergence. Thus we obtain

(4) $$\mathrm{Re}\{-\,i\,E\} = \int_0^k -\,i(1 + e^{-2\mu h})\,\frac{\lambda^3\,d\lambda}{|\mu|} + 2\,\mathrm{Re}\Big\{i\int_0^\infty e^{-2\mu h}\,\frac{\mu_E}{n^2\,\mu + \mu_E}\,\frac{\lambda^3\,d\lambda}{\mu}\Big\}.$$

If we take the values of these integrals given by (36.13) to (36.17), substitute (4) in (1), and append the factor (36.22) in order to express the result in terms of our units, then we obtain the value of W from (36.23).

b) *Horizontal antenna.* For $\Pi = (\Pi_x, \Pi_z)$ and $\mathbf{E} = \mathbf{E}_x$ we obtain from (31.4):

(5) $$E = k^2\,\Pi_x + \frac{\partial^2 \Pi_x}{\partial x^2} + \frac{\partial^2 \Pi_z}{\partial x\,\partial z}.$$

Now according to (33.12) and (33.15) the x-dependence of Π_x is given by $I_0(\lambda\,r)$, and the x-dependence of Π_z is given by $\frac{x}{r}\,I_1(\lambda\,r)$. Hence for small x,y we have

$$I_0(\lambda r) = 1 - \frac{\lambda^2}{4}(x^2 + y^2) + \cdots, \qquad \frac{x}{r} I_1(\lambda r) = \frac{\lambda}{2} x + \cdots,$$

and for $r = 0$

$$\left(k^2 + \frac{\partial^2}{\partial x^2}\right) I_0 = \frac{1}{2}(2k^2 - \lambda^2), \qquad \frac{\partial}{\partial x}\frac{x}{r} I_1 = \frac{\lambda}{2}. \tag{6}$$

These factors $\frac{1}{2}(2k^2 - \lambda^2)$ and $\lambda/2$ appear under the integral signs of the equations (33.12) and (33.15) in the computation of (5), where for the first two terms on the right side of (33.12) we have to use (31.4) and where in (33.15) we have to perform the differentiation with respect to z, in addition to the differentiation with respect to x (this yields a factor $-\mu$ under the integral sign). Thus, instead of (5) we obtain

$$\begin{aligned} E = {} & \frac{1}{2}\int_0^\infty \frac{2k^2 - \lambda^2}{\mu}\left(e^{-\mu|z-h|} - e^{-\mu(z+h)}\right)\lambda^2\, d\lambda \\ & + \int_0^\infty e^{-\mu(z+h)}\left[\frac{2k^2 - \lambda^2}{\mu + \mu_E} + \frac{\lambda^2}{k^2}\frac{\mu(\mu - \mu_E)}{n^2\mu + \mu_E}\right]\lambda\, d\lambda. \end{aligned} \tag{7}$$

The first term in [ ] is due to the third term in (33.12), the second term is due to (33.15). If we form the common denominator of [ ] and observe that $\mu^2 - \mu_E^2 = k^2(n^2 - 1)$ (see p. 260), then we obtain

$$[\,] = \frac{\lambda^2 - 2\mu\mu_E}{n^2\mu + \mu_E}.$$

Hence for $z = 0$ the second line of (7) becomes identical with the integral for L in (36.17a). If in the first line of (7) we pass to $\mathbf{Re}\,\{-i\,E\}$ according to the procedure of (3), then we can again replace the upper limit ∞ by k and carry out the integration as in (36.16a). If we then pass to our system of units we obtain exactly the expression in (36.23a).

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